Department of Mathematical Sciences
Division of Applied Mathematics
Stellenbosch University

# 2019 | MS763 | Applied Markov Processes

## Problems for Chapter 5: Stochastic differential equations

Due: 25 October 2019

### Theoretical

**Q1.** **(Geometric Brownian motion)** Consider the drifted Brownian motion $X_t = \mu t + \sigma W_t$, where $\mu$ is the drift, $\sigma$ the noise amplitude, and $W_t$ is a Brownian motion or Wiener process. We can also express $X_t$ in terms of the SDE

$$dX_t = \mu dt + \sigma dW_t. \tag{1}$$

(a) Consider a new process $Z_t$ defined as $Z_t = e^{X_t}$. This is a transformation of drifted Brownian motion. Using *normal* calculus rules, derive the SDE satisfied by$Z_t$.

(b) Use Itô's calculus to derive the correct SDE satisfied by $Z_t$.

(c) Taking $Z_t$ to represent the price (in US$) of Microsoft's stock in time, calculate the probability that the price goes above $150 at $t = 10$. Express your answer in terms of $\mu$, $\sigma$, and $Z_0$.

**Q2.** **(Gradient diffusions)** Consider the SDE in $\mathbb{R}^d$ defined by

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t, \tag{2}$$

where $U(x)$ is a smooth function, called the potential, $\sigma$ is the noise amplitude (real, positive), and $W_t$ is a $d$-dimensional Brownian motion. Assume that $U(x)$ is convex (U-shaped), and so that it has a unique minimum at some point $x^*$. Assume also that $X_t$ is ergodic, so there exists a unique stationary distribution $p^*$.

(a) Consider the SDE *without* noise by setting $\sigma = 0$. What is the long-time behavior of the corresponding ODE?

(b) Find the expression of the stationary distribution $p^*$ with $\sigma > 0$ by solving the Fokker-Planck equation.

(c) Discuss the shape of $p^*$ in relation to $x^*$. Where does $p^*$ concentrate as $\sigma \to 0$?

### Numerical

**Q3.** **(Stochastic gradient descent)** We seek to find the global minimum of

$$U(x) = \frac{x^4}{2} - 5x^2 + x. \tag{3}$$

This potential has a positive local minimum in addition to its global minimum, which is negative.

(a) Find numerically the positions of the local and global minima of $U(x)$ using any routine or function in R, Python, Matlab or Mathematica.

(b) Solve the gradient descent dynamics, defined by the ODE

$$\dot{x}(t) = -U'(x(t)), \tag{4}$$

for various initial conditions. You can use `ode23` in Matlab, `odeint` in Python or your own discretization scheme. Analyse your results in view of locating the global minimum of $U(x)$.

(c) Solve the stochastic gradient descent dynamics, defined by the SDE

$$dX_t = -U'(X_t)dt + \sigma dW_t, \tag{5}$$

for various initial conditions and noise amplitudes $\sigma$ using the Euler–Maruyama scheme. Analyse your results and compare them with part (b). Does $X_t$ always reach the global minimum? [Note: Use $T = 10$ and $\sigma = 0.5$, then try $T = 100$ and $\sigma = 0.25$.]

(d) Repeat part (c), but now decrease the noise in time according to $\sigma_t = \frac{\alpha}{t+1}$. Try $\alpha \approx 1$ and $T \approx 10$ to 100 to see if you can locate the global minimum. [Note: Decreasing $\sigma$ in time is referred to as *annealing* or *stochastic relaxation*.]

(e) What is the advantage of stochastic gradient descent over deterministic gradient descent?

**Q4. (Kapitza pendulum)** The Kapitza ODE

$$\ddot{\theta}(t) + [1 + A\cos(\omega t)]\sin\theta(t) + k\dot{\theta}(t) = 0 \tag{6}$$

models the evolution of a simple pendulum with friction vibrated at its base. We can model random vibrations by adding Gaussian white noise $\xi(t) = dW(t)/dt$ to this equation to obtain

$$\ddot{\theta}(t) + [1 + A\cos(\omega t)]\sin\theta(t) + k\dot{\theta}(t) = \sqrt{\eta}\,\xi(t). \tag{7}$$

The parameter $\eta$ is the amplitude of the noise.

(a) Simulate the Kapitza model *without* noise and show that the upright position $\theta = \pi$ is stable using $A = 20$, $\omega = 10$, $k = 0.5$ and initial values $\theta(0) = 3\pi/4$ and $\dot{\theta}(0) = 0$. You can use again `ode23` in Matlab, `odeint` in Python or your own discretization scheme. [Note: You will have to transform the 2nd order ODE to a set of 1st order ODEs.]

(b) Simulate trajectories of the Kapitza model now *with* noise using the Euler-Maruyama scheme. Use the same parameters as before and $\eta$ in the range $[0.001, 0.05]$. Also integrate up to $T = 50$ with $\Delta t = 0.05$.

(c) Increase $\eta$ to find, very approximately, the noise threshold at which the pendulum cannot be stabilised in its upright position anymore.

## Reading

- GS, Secs. 13.7 and 13.8 on stochastic (Itô) calculus.
- D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Review 43, 525, 2001. (pdf available on SunLearn).