Department of Mathematical Sciences
Division of Applied Mathematics
Stellenbosch University

2019 | MS763 | Applied Markov Processes

Problems for Chapter 1: Probability theory and sampling

Due: 13 August 2019

Theoretical

Q1. **(Random fraction)** Find the probability that a randomly written fraction will be irreducible. Consider that the numerator and denominator are randomly selected numbers from the sequence $1, 2, \dots, k$ and let $k \to \infty$. [Prize question in 1st year probability course.]

Q2. **(Common random variables)** Calculate the mean, variance, characteristic function, and generating function of the following RVs:

(a) Bernoulli p: $P(X = 1) = p$, $P(X = 0) = 1 - p$.

(b) Binomial with parameters (n, p).

(c) Gaussian $X \sim \mathcal{N}(\mu, \sigma^2)$.

(d) Exponential:
$$p(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \quad \lambda > 0.$$

(e) Uniform over $[0, L]$.

(f) Cauchy (also known as Lorentzian by physicists) with scale σ:
$$p(x) = \frac{1}{\pi} \frac{\sigma}{x^2 + \sigma^2}, \qquad x \in \mathbb{R}, \quad \sigma > 0.$$

For that one, calculate only the mean and variance.

Q3. **(Odd fish)** Show for a Poisson RV X that $P(X \text{ is odd}) < P(X \text{ is even})$.

Q4. **(Binomial limits)** Consider a binomial RV X with parameters (n, p).

(a) Show that X converges to a Poisson RV in the limit $n \to \infty$ with $np = \text{constant}$. [For the physicists: How does that relate to radioactivity?]

(b) Show that X converges (when properly standardised) to a Gaussian RV in the limit $n \to \infty$ with p fixed. [DeMoivre-Laplace limit.]

Q5. **(Sums of random variables)**

(a) Write down the probability density of $Z = X + Y$ for two independent RVs X and Y as a convolution integral.

(b) Show using your result of (a) that the sum of two IID Gaussian RVs is Gaussian-distributed.

Q6. **(Rayleigh distribution)** Let X, Y be two independent Gaussian RVs with mean 0 and variance 1. Show that $\theta = \arctan(Y/X)$ is uniform over $[-\pi/2, \pi/2]$ and that $R = \sqrt{X^2 + Y^2}$ is distributed according to the Rayleigh function:
$$p(r) = r e^{-r^2/2}, \quad r \geq 0.$$

[Hint for θ: Find the pdf of Y/X first and then the pdf of $\arctan(Y/X)$ by transformation of RVs.]

Q7. **(Log-normal RVs)** Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and define $Y = e^X$. Find the probability density of Y.

Numerical

Q8. **(Histogram function)** Construct a function called `histogram(v,a,b,n)` that constructs a histogram of the values contained in the vector `v` in n bins spread uniformly in the interval $[a, b]$. The output of your function is the vector of histogram counts. Show that your function works correctly by comparing it with the similar histogram function available in Matlab, R or Python.

Q9. **(Non-uniform variates)** Use the transformation of RVs method to construct random number generators for the following probability distributions and test them with large-enough samples by plotting the sample histogram (properly normalised) with the corresponding theoretical distribution.

(a) Bernoulli p.

(b) Uniform over $[-1, 1]$.

(c) Exponential with parameter λ.

(d) Cauchy distribution with center 0 and scale σ.

(e) Rayleigh distribution with parameter r.

Q10. **(Box-Muller method)** Code the Box-Muller method (read about it first) and show that it works by performing histogram analyses on large-enough samples.

Q11. **(Point estimation)** Describe a procedure (any) to estimate the parameter λ of a sample of exponentially-distributed RVs. Show that your procedure works with a numerical example.

Q12. **(Monte Carlo estimation of π)** Choose a point (x, y) at random in the square

$$S = \{(x, y) : x \in [-1, 1] \text{ and } y \in [-1, 1]\}.$$

Show that the probability that the point lies in the circle

$$C = \{(x, y) : x^2 + y^2 = 1\}$$

is equal to $\pi/4$. Turn this result into a computer program that gives an approximation of π. Show that it works, with error bars. [Another prize question from the 1st year probability course.]

Prize question

R100 for the best complete answer. Hand in your solution on a separate sheet.

We know from Q5 that a sum of Gaussian RVs is Gaussian-distributed. Are there other types of RVs that have this 'stable' property, that is, that satisfy $Z \sim X \sim Y$ for $Z = X + Y$ and X, Y independent? Limit yourself to symmetric RVs with even probability densities.