Large deviations in Statistical Physics
Week 1 coursework: Elements of probability theory

Q1. (Practice and revision) Go over all the examples in the week-1 notes.

Q2. (Common random variables) Calculate the mean, variance, characteristic function, and generating function of the following random variables (RVs):

(a) Bernoulli: $X = 0$ with probability $1 - p$ and $X = 1$ with probability p.

(b) Gaussian: $X \sim \mathcal{N}(\mu, \sigma^2)$, that is,

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$

(c) Exponential:

$$p_X(x) = \mu e^{-\mu x}, \quad x \geq 0.$$

(d) Uniform: $X \sim \mathcal{U}[0, L]$, that is,

$$p_X(x) = \begin{cases} 1/L & x \in [0, L] \\ 0 & \text{otherwise.} \end{cases}$$

(e) Cauchy:

$$p_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad x \in \mathbb{R}.$$

You can verify your answers with Wikipedia and some of the examples of the notes. Explain the result of the generating function for (e).

Q3. (Sums of random variables)

(a) Write down the probability density function (pdf) of $Z = X + Y$ for two independent RVs X and Y as a convolution integral.

(b) Show using your result of (a) that the sum of two IID Gaussian RVs is Gaussian distributed.

(c) Repeat for two Cauchy random variables.

Q4. (Rayleigh distribution) Let X, Y be two independent Gaussian RVs with mean 0 and variance 1. Show that $\theta = \arctan(Y/X)$ is uniform over $[-\pi/2, \pi/2]$ and that $R = \sqrt{X^2 + Y^2}$ is distributed according to the Rayleigh pdf:

$$p(R = r) = r e^{-r^2/2}, \quad r \geq 0.$$

[Hint for θ: Find the pdf of Y/X first and then the pdf of $\arctan(Y/X)$ by transformation of RVs.]