6.1. Recap on Markov chains

$$X_1 \xrightarrow[\pi]{} X_2 \to X_3 \to \cdots \to X_n \qquad , \; X_i \in \Lambda = \{1, 2, \ldots, q\}$$

· Transition probabilities

$$\pi(j|i) = P(X_n = j \mid X_{n-1} = i) \quad \text{assumed } \overset{\text{time}}{\text{homogeneous}}$$
$$= P(i \to j)$$
$$= P(j|i)$$

· Transition matrix : Π : $\quad \Pi_{ji} = \pi(j|i)$ ↑ row ↑ column

· Stochastic matrix :

$$\sum_{j \in \Lambda} \pi(j|i) = 1 \quad \forall\, i \qquad \Rightarrow \quad \vec{1}\,\Pi = \vec{1}$$

where $\vec{1} = (1, 1, \ldots, 1)$

- Propagation (Master equation):

$$\vec{p}_n = \Pi\, \vec{p}_{n-1}$$

i.e.

$$p_n(j) = \sum_i \pi(j|i)\, p_{n-1}(i)$$

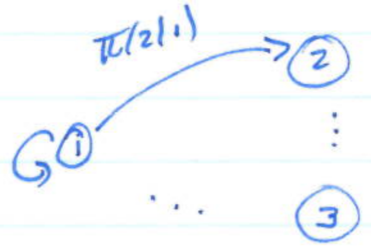
- Stationary distribution:

$$\vec{p}^* = \lim_{n\to\infty} \vec{p}_n = \lim_{n\to\infty} \Pi^{n-1} \vec{p}_1 \qquad \text{ergodic Markov chain}$$

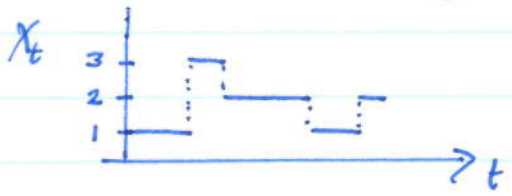
$$\vec{p}^* = \Pi\, \vec{p}^* \qquad \text{fixed point of } \Pi$$

eigenvector with eigenvalue 1.

- Graphical representation :



6.2. Continuous-time (jump) processes



$X_t \in \Lambda = \{1, 2, \dots, 9\}$

- Transition probability:

$$\Pi_{ji}(s,t) = P(X_t = j \mid X_s = i) \qquad t \geq s$$

- Homogeneous (stationary) case:

$$\Pi_{ji}(t) = P(X_{t+s} = j \mid X_s = i) \qquad \text{ind. of } s$$

propagator

$$= P(X_t = j \mid X_0 = i)$$
$$= P(i \to j \text{ in time } t)$$

- Transition matrix: $\Pi(t) \quad : \quad \Pi_{\underset{\text{row}}{j}\,\underset{\text{column}}{i}}(t)$

- $t=0$: $\Pi(0) = \mathbb{1}$ no transitions, no evolution
- Stochastic matrix : $\sum_{j\in\Lambda} \Pi_{ji}(t) = 1 \quad \forall\, i,\ t \geq 0$

$$\Rightarrow \quad \vec{1}\,\Pi(t) = \vec{1}$$

- Propagation (Master equation) :

$$p_j(t) = P(X_t = j) = \sum_i \Pi_{ji}(t)\, p_i(0)$$

$$\vec{p}(t) = \Pi(t)\,\vec{p}(0)$$

- Infinitesimal propagator :

$$\Pi(\Delta t) = \Pi(0) + G\,\Delta t + O(\Delta t^2)$$
$$= \mathbb{1} + \underbrace{G}_{\text{Generator}}\Delta t + O(\Delta t^2)$$

$$G = \lim_{\Delta t \to 0} \frac{\Pi(\Delta t) - \Pi(0)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Pi(\Delta t) - \mathbb{1}}{\Delta t}$$

$$\Rightarrow \quad \Pi(\Delta t) = e^{G\Delta t}$$

- Off-diagonal part:

$$\Pi_{ji}(\Delta t) = G_{ji}\,\Delta t = \underbrace{W(i \to j)}_{\substack{\text{transition rate} \\ \text{prob / unit time}}}\,\Delta t \qquad i \neq j$$

- Diagonal part:

$$\Pi_{ii}(\Delta t) = 1 + G_{ii}\,\Delta t$$

- Normalization:

$$\sum_j \Pi_{ji}(\Delta t) = \Pi_{ii}(\Delta t) + \sum_{j \neq i} \Pi_{ji}(\Delta t)$$

$$= 1 + G_{ii}\Delta t + \sum_{j \neq i} G_{ji}\,\Delta t \qquad = 1$$

$$\Rightarrow \quad G_{ii} + \sum_{j \neq i} G_{ji} = 0 \qquad \text{or} \qquad \sum_j G_{ji} = 0$$

columns sum to zero

- G_{ii} is unspecified, so take

$$-G_{ii} = \sum_{j \neq i} G_{ji} = r_i$$

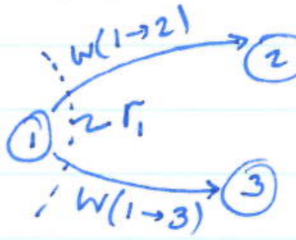
escape rate from i

- Complete form:

escape rate in diagonal

$$G_{ji} = \underbrace{W_{ji}}_{\substack{\text{off diagonal} \\ \text{transition rate}}} - r_i\,\delta_{ij}$$

- Graphical representation:



- Propagation (Master equation):

$$\frac{d}{dt}\vec{p}(t) = \underbrace{G}_{\text{generator}}\,\vec{p}(t)$$

i.e.

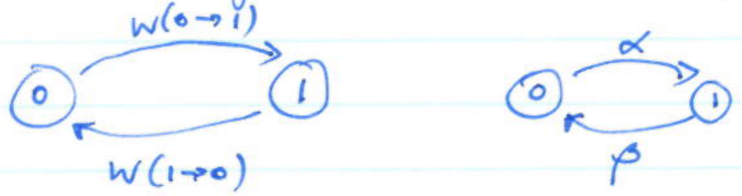
$$\frac{d}{dt} p_j(t) = \sum_{i \neq j} \Big(\underbrace{G_{ji}\,p_i(t)}_{\text{flow coming from } i} - \underbrace{G_{ij}\,p_j(t)}_{\text{flow leaving from } j} \Big)$$

- Stationary distribution:

$$\frac{d}{dt}\vec{p}^{*} = 0 = G\vec{p}^{*}$$

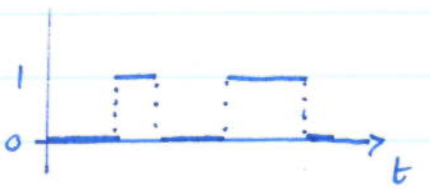
$$\Rightarrow \quad G\vec{p}^{*} = 0 \qquad \text{eigenvector of eigenvalue } 0$$

Example: 2 state jump process $\Lambda = \{0, 1\}$



no self transit
no need to specify self transition

$$G = \begin{pmatrix} & w(1\to 0) \\ w(0\to 1) & \end{pmatrix} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix}$$



- Time spent in one state is exponentially-distributed:

$$P\left(X_t = i \text{ for } t \in [0,\tau]\right) = P\left(X_t = i \text{ for } t \in [0,\Delta t)\right) \cdots P\left(X_t = i \; t \in [\tau-\Delta t, \tau]\right)$$

$$= P(\text{no transition in } \Delta t)^{\tau/\Delta t}$$

$$= \left(\Pi_{i,i}(\Delta t)\right)^{\tau/\Delta t}$$

$$= (1 - r_i \Delta t)^{\tau/\Delta t} \qquad n = \frac{\tau}{\Delta t}$$

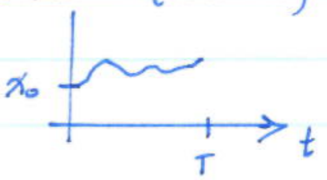
$$= \left(1 - r_i \frac{\tau}{n}\right)^n$$

$$\to e^{-r_i \tau}$$

6.3 Ordinary differential equations (ODEs)

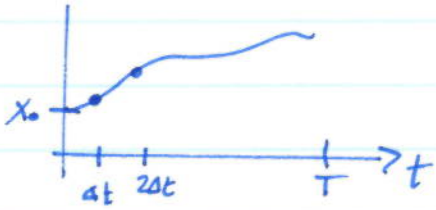
$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

initial condition "force"



- Euler discretization:

$$\begin{cases} x(t+\Delta t) = x(t) + f(x(t),t)\,\Delta t \\ x(0) = x_0 \end{cases}$$



or

$$\begin{cases} \Delta x(t) = f(x(t), t)\,\Delta t \\ x(0) = x_0 \end{cases}$$

- Matlab code:

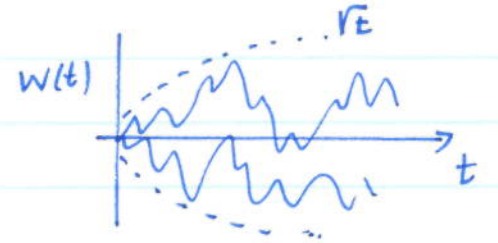
```
T=5.0;
dt = 0.01;
n = floor(T/dt);
x = zeros(1, n+1);
x(1) = 4.0;

for i=1:n
    x(i+1) = x(i) + f(x(i))*dt ;
end

tspan = [0:dt:T];
plot(tspan,x);
```

6.4. Brownian motion or Wiener motion

$W(t)$, $t \geq 0$ such that

- $W(0) = 0$
- $W(t) \sim \mathcal{N}(0, t) \Rightarrow E[W(t)] = 0 \quad \forall\, t$

  $\mathrm{var}\, W(t) = t$



- Wiener increment:
  - $W(t) - W(s) \sim \mathcal{N}(0, t-s) \qquad t > s$
  - Independent increments

- $W(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{\Delta W_n}_{\sim \mathcal{N}(0, \Delta t)}$

$n = t/\Delta t$

- $\Delta W(t) \sim \mathcal{N}(0, \Delta t) = \sqrt{\Delta t}\, \mathcal{N}(0,1)$

6.5. Stochastic differential equations (SDEs)

ODE

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases}$$

initial condition

SDE

$$\begin{cases} \dot{X}(t) = f(X(t)) + \underbrace{\xi(t)}_{\text{noise}} \\ X(0) = x_0 \end{cases}$$

rv

- Noise model: $\xi(t) = \frac{dW(t)}{dt}$ Gaussian white noise
- Discretization:

$$\begin{cases} x(t+\Delta t) = x(t) + f(x(t))\Delta t \\ x(0) = x_0 \end{cases}$$

$$X(t+\Delta t) = X(t) + f(X(t))\Delta t + \underbrace{\Delta W(t)}_{\sim \mathcal{N}(0, \Delta t)}$$

Wiener increment

$$dx(t) = f(x(t))\, dt$$

$$dX(t) = f(X(t))\, dt + \underbrace{dW(t)}_{\sim \mathcal{N}(0, dt)}$$

Euler

Euler-Maruyama

Examples:

1- Brownian motion: $dX(t) = dW(t)$ $f=0$
$\Rightarrow X(t) = W(t)$

2- Drifted Brownian motion: $dX(t) = \underbrace{\mu}_{\text{drift}} dt + \underbrace{\sigma}_{\text{volatility}} dW$

3- Langevin equation or Ornstein-Uhlenbeck process:
$$dX(t) = \underbrace{-a\,X(t)}_{\text{linear force}}\,dt + \sigma\,dW(t)$$

4- Geometric Brownian motion:
$$dS(t) = \mu\,S(t)\,dt + \sigma\,S(t)\,dW(t)$$

Simulation:

```
T= 5.0;
dt = 0.01;
n = floor(T/dt);
x = zeros(1, n+1);
x(1) = 4.0;
sigma = 1.0;
for i = 1:n
    x(i+1) = x(i) + f(x(i))*dt + sigma*sqrt(dt)*randn;
end

tspan = [0 : dt : T];
plot(tspan, x);
```

- Infinitesimal propagator:

$$dX(t) = f(X(t))\,dt + \sigma\, dW(t)$$

$$X(t+\Delta t) = X(t) + f(X(t))\,dt + \sigma \underbrace{dW(t)}_{\sim \mathcal{N}(0,dt)}$$

$$x' = x + f(x) + \sigma \underbrace{Z}_{\sim \mathcal{N}(0,dt)}$$

Transformation of Gaussian RV:

$$\Pi_{\Delta t}(x'|x) = P(X_{t+\Delta t} = x' \mid X_t = x) \qquad \text{Gaussian}$$

$$\downarrow$$

$G(x'|x)$ generator

- Fokker-Planck equation:

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\Big(f(x)\,p(x,t)\Big) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} p(x,t)$$

$$= L\, p(x,t)$$

linear operator = FP operator

- L = dual of G

$$= -\frac{\partial}{\partial x} f + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$$

$$\Rightarrow \quad G = L^{\dagger} = f\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} \qquad \left(\left(\frac{\partial}{\partial x}\right)^{\dagger} = -\frac{\partial}{\partial x}\right)$$

- Stationary distribution:

$$\frac{\partial}{\partial t} p(x)^{*} = 0$$

$$\Rightarrow \quad L p^{*} = 0$$