3.1. Varadhan's theorem

Ref: S.R.S. Varadhan, Comm. Pure & Appl. Math, 1966
{ Abel Prize 2007

· Exponential expectation:

$$E[e^{nf(A_n)}] = \int e^{nf(a)} P(A_n=a)\, da \quad \sim e^n$$

· Limit function:

$$\lambda[f] = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nf(A_n)}]$$

functional of f

· Assumptions:

1. A_n satisfies the LDP: $P(A_n=a) \asymp e^{-nI(a)}$
2. f bounded (can be relaxed)

Theorem:

$$\lambda[f] = \sup_a \{ f(a) - I(a) \}$$

"Proof":

$$\begin{aligned} E[e^{nf(A_n)}] &= \int e^{nf(a)} P(A_n=a)\, da \\ &\asymp \int e^{nf(a)} e^{-nI(a)}\, da \\ &= \int e^{n\{f(a)-I(a)\}}\, da \\ &\asymp e^{n \max\{f(a)-I(a)\}} \end{aligned}$$

exponential integral

Laplace principle or approximation

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nf(A_n)}] = \max\{f(a) - I(a)\}$$

$\boxtimes$

Rem :

- Diff between max and sup

  $\sup [a,b) = b$ max is undefined

  $\sup [a,\infty) = \infty$

  ~~max~~

- A_n is any RV : real, vector, function, etc.

- Varadhan's result is a generalization of Laplace's approximation (to general spaces)

Particular case: $f(a) = ka$ $k \in \mathbb{R}$ (not bounded)

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkA_n}]$$ real function

$$\Rightarrow \lambda(k) = \sup_a \{ka - I(a)\}$$ Legendre-Fenchel transform

[~~Questions / Get I from λ?~~

Example: $S_n = \frac{1}{n}\sum_{i=1}^n X_i$ $X_i \sim N(\mu,\sigma^2)$

$$I(s) = \frac{(s-\mu)^2}{2\sigma^2}$$

$$\lambda(k) = \sup_s \{ks - I(s)\}$$

$$\partial_s\left(ks - I(s)\right) = 0 \Rightarrow k = I'(s) = \frac{s-\mu}{\sigma^2}$$

$$s^* = \sigma^2 k + \mu$$

$$\begin{aligned}\lambda(k) &= ks^* - I(s^*)\\ &= k(\sigma^2 k + \mu) - \frac{(\sigma^2 k + \mu - \mu)^2}{2\sigma^2}\\ &= \sigma^2 k^2 + k\mu - \frac{\sigma^2 k^2}{2}\\ &= k\mu + \frac{\sigma^2}{2}k^2\end{aligned}$$

LF parabola = parabola

Compare with direct result.]

$$E[e^{nkS_n}] = E[e^{k\Sigma_i X_i}] = E[e^{kX}]^n$$

$$\Rightarrow \lambda(k) = \ln E[e^{kX}]$$

But $E[e^{kX}] = e^{k\mu + \frac{\sigma^2}{2}k^2}$

$$\Rightarrow \lambda(k) = k\mu + \frac{\sigma^2}{2}k^2$$

Question: Get I from λ?

3.2. Gärtner-Ellis Theorem

Refs: J. Gärtner, Th. Prob. Appl. 1977

R.S. Ellis, Ann. Prob., 1984

· Scaled cumulant generating function:

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E\left[e^{nkA_n}\right] \qquad k \in \mathbb{R}.$$

· Assumptions:

- $\lambda(k) < \infty$
- $\lambda(k)$ is differentiable ~~$\forall k \in \mathbb{R}$~~ (Gärtner's version)

· <u>Theorem</u>:

① A_n satisfies the LDP: $P(A_n = a) \asymp e^{-nI(a)}$

② $I(a) = \sup_{k\in\mathbb{R}} \{ka - \lambda(k)\}$ Legendre-Fenchel transform

"Proof": cf HT2009, Appendix C

$$P(A_n = a) = \frac{1}{2\pi i}\int_B e^{-ka}\, M_{A_n}(k)\, dk$$

$$\asymp \int_B e^{-nka}\, e^{n\lambda(k)}\, dk$$

$$= \int_B e^{-n(ka - \lambda(k))}\, dk$$

$$\asymp e^{-n \min_k \{ka - \lambda(k)\}}$$ along contour

$$= e^{-n \max_k \{ka - \lambda(k)\}}$$ perpendicular to contour for $k \in \mathbb{R}$

cf Appendix

$$\Rightarrow \lim_{n\to\infty} -\frac{1}{n} \ln P(A_n = a) = I(a) = \max_k \{ka - \lambda(k)\}$$

Rem:

- Why $\lambda(k)$ differentiable? See next week
- An not necessarily iid sum – any RVs.

Example:

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad X_i \sim \mathcal{N}(\mu, \sigma^2) \text{ iid}$$

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkS_n}] = \ln E[e^{kX}]$$

$$= k\mu + \frac{\sigma^2}{2} k^2$$

Differentiable $\forall\, k \in \mathbb{R}$.

$$\Rightarrow P(S_n = s) \asymp e^{-nI(s)}$$

$$I(s) = \sup_k \{ks - \lambda(k)\}$$

$$\partial_k (ks - \lambda(k)) = 0 \Rightarrow s = \lambda'(k) = \mu + \frac{\sigma^2}{\cancel{2}} k$$

$$k^* = \frac{s-\mu}{\sigma^2}$$

$$\Rightarrow I(s) = k^* s - \lambda(k^*)$$

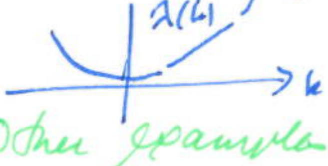
$$= \left(\frac{s-\mu}{\sigma^2}\right) s - \left(\frac{s-\mu}{\sigma^2}\right)\mu + \frac{\sigma^2}{2} \frac{(s-\mu)^2}{\sigma^4}$$

$$= \frac{(s-\mu)}{\sigma^2}(s-\mu) + \frac{(s-\mu)^2}{2\sigma^2}$$

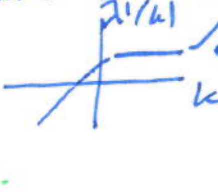
$$= \frac{(s-\mu)^2}{2\sigma^2}$$

Inverse calculation compared to last calculation.
LF parabola = parabola

Rem: Why Legendre-Fenchel and not Legendre transform?



$\lambda'(k) = s$



← solution k^* ?

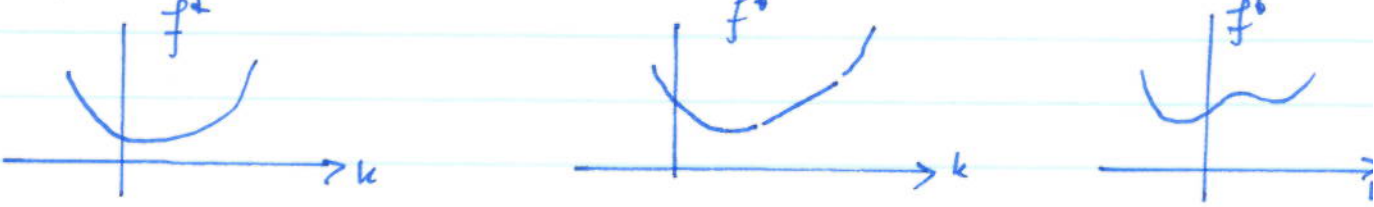
Other examples next week.

3.3. Legendre - Fenchel transforms

$f : \mathbb{R} \to \mathbb{R}$

<u>Def</u>: $f^*(k) = \sup_x \{ kx - f(x) \}$ $\qquad k \in \mathbb{R}$

- $f^*(k)$ is a convex function



convex $\qquad$ convex, non-strictly $\qquad$ non-convex

- Inverse LF transform: If $f(x)$ is convex, then

$$f(x) = \sup_k \{ kx - f^*(k) \}$$

- LF is self-inverse (dual or involutive) for convex functions

- Legendre transform: If $f(x)$ is differentiable and strictly convex, then

$$f^*(k) = k\, x_k^* - f(x_k^*) \qquad f'(x) = k$$

Proof: f diff: $\quad \partial_x \left(kx - f(x) \right) = 0 \quad \Rightarrow f'(x) = k$

f strictly convex: $f'(x) = k$ has unique root x_k^* $\forall k \in \mathbb{R}$

$\Rightarrow f^*(k) = k x_k - f(x_k)$. ∎

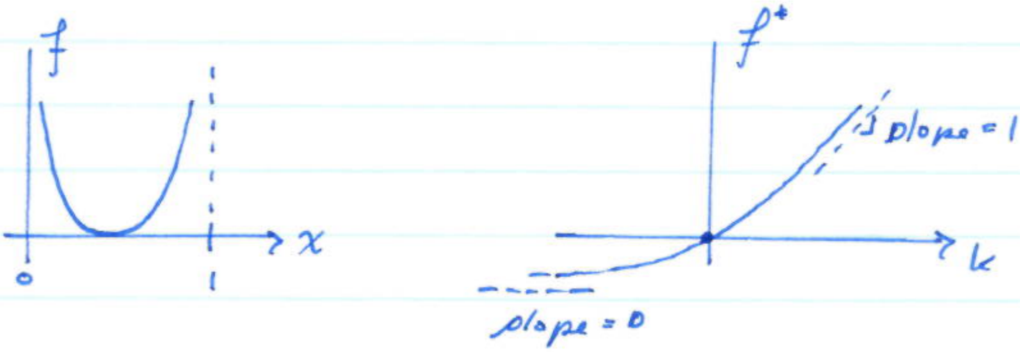
Rem: LF more general because can be applied to nondifferentiable and/or nonconvex functions.

• Duality:

$$f' \leftrightarrow k$$

$$f^{*\prime} \leftrightarrow x$$

Example:



~~Example~~:

Example: $f(x) = \frac{(x-a)^2}{2b}$ $\quad f^*(k) = ak + \frac{b}{2}k^2$

The only fix point.

3.4. Properties of $\lambda(k)$ and $I(s)$

• $\lambda(0) = k$

Proof: $\lambda(k) = \lim\limits_{n\to\infty} \frac{1}{n} \ln E[e^{nkS_n}]$

$$\lambda(0) = \lim \frac{1}{n} \ln E[\underbrace{1}_{=1}]$$

iid case: $\lambda(k) = \ln E[e^{kX}]$

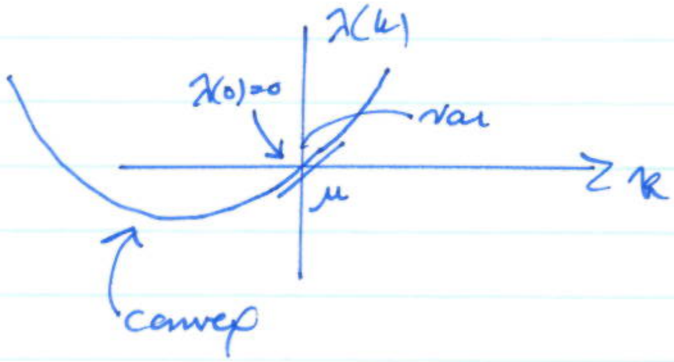
$$\lambda(0) = 0$$

• $\lambda'(0) = E[X]$ iid case
• $\lambda''(0) = var(X)$ iid case

• $\lambda'(0) = \lim\limits_{n\to\infty} E[S_n]$ in general

• $\lambda(k)$ is convex
Exercise

• Rate functions obtained from GE are convex.
Rem: $I(s)$ may be nonconvex
Example next week



More from exercises