Classical and quantum processes with random resetting

Hugo Touchette

Department of Mathematical Sciences
Stellenbosch University, South Africa

NITheP/CS, UKZN talk, South Africa

Plan

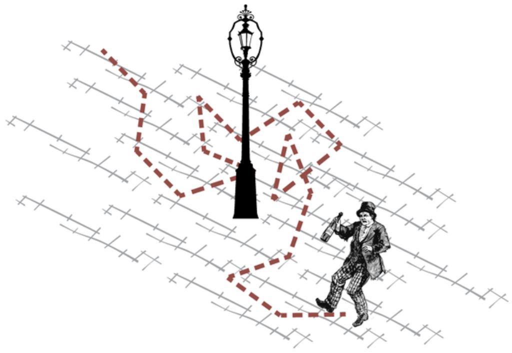
- Markov processes
- Processes with reset
- Project 1: Spectral properties
- Project 2: Large deviations
- Examples

Work with

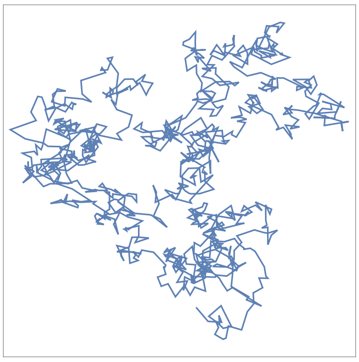
- Janusz Meylahn (MSc SU now in Leiden)
- Sanjib Sabhapandit (Raman Institute, India)
  [Meylahn, Sabhapandit, HT PRE 2015]
- Dominic Rose, Igor Lesanovsky, Juan Garrahan (University of Nottingham)
  [Rose, HT, Lesanovsky, Garrahan PRE 2018]

Reset processes

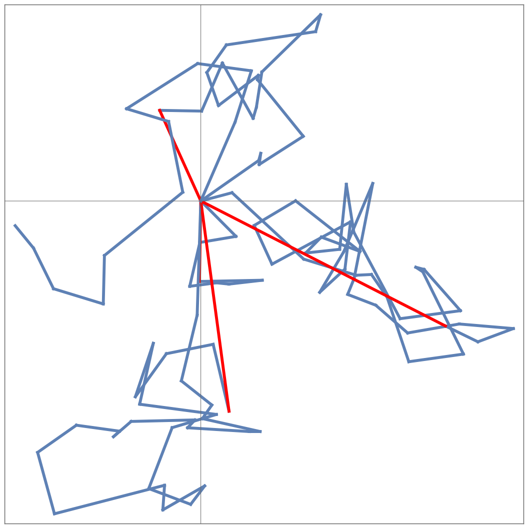
Drunkard's walk



2D Brownian motion (BM)



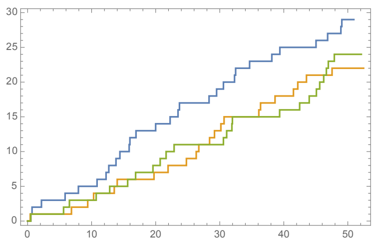
Random walk with resets



- BM: $x_{n+1} = x_n + \sqrt{\Delta t}\ \texttt{randn()}$
- RBM: if $\texttt{rand()} < \Gamma \Delta t$ then reset
- Poisson process for resets

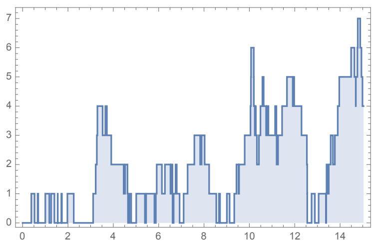
Applications

Population models



- Random births and deaths
- Reset $=$ decimation

Queueing models



- Random arrivals and service
- Reset $=$ random clearing

Variants

- Reset at fixed point
- Reset at random point
- Different waiting times
- Non Poisson resets
- Many-particle process
- Non-instantaneous resets
- ...

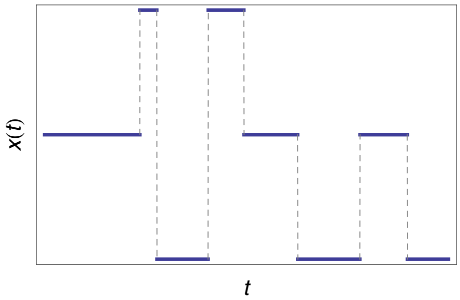
Superposition of processes
Compound processes
Switching processes

Markov processes

- State: X_t
- Probability distribution: $p(x,t)$
- Master equation:

$$\partial_t p(x,t) = \mathcal{L} p(x,t)$$

- Ergodic: $\mathcal{L} p_{ss} = 0$

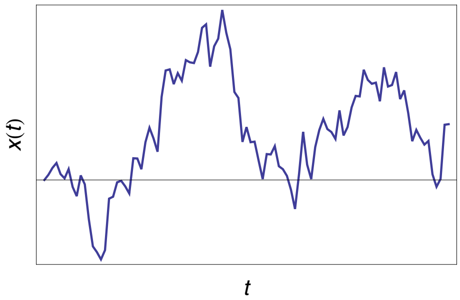


Jump processes

- Discrete states, continuous time
- Queues, population models

Diffusions

- Continuous state, continuous time
- Noise perturbed dynamical systems



Markov processes (cont'd)

Jump processes

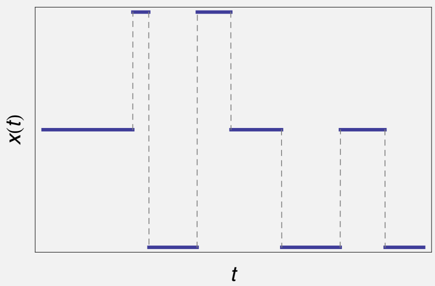
- Transition rates:

$$W(C \to C') = P(C \to C' \text{ in } dt)/dt$$

- Escape rates:

$$R(C) = \sum_{C'} W(C \to C')$$

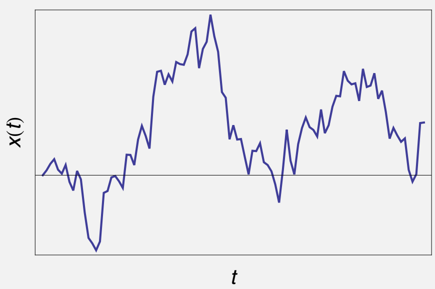
- Generator: $\mathcal{L} = \underbrace{W}_{\text{off-diag}} - \underbrace{RI}_{\text{diag}}$



Diffusions

- SDE: $dX_t = F(X_t)dt + \sigma dW_t$
- Generator:

$$\mathcal{L} = -\nabla \cdot F + \frac{D}{2}\nabla^2, \qquad D = \sigma\sigma^T$$



Reset processes

Jump processes

- Reset state: C_0
- Reset rate Γ: $P(C \to C_0 \text{ in time } dt) = \Gamma dt$

Diffusions

- Reset state: x_0
- SDE + Poisson process $x \to x_0$

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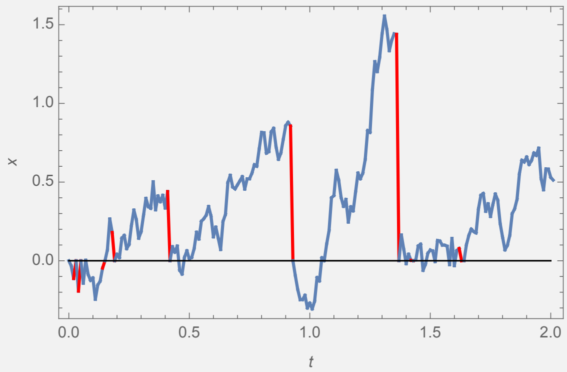
if rand()<Gamma*dt:
        x=xr
else:
        x=x+F(x)*dt+sigma*sqrt(dt)*randn()
```



- Fokker-Planck equation:

$$\partial_t p(x,t) = \mathcal{L}p(x,t) - \underbrace{\Gamma\, p(x,t)}_{\text{sink}} + \underbrace{\Gamma\, \delta(x-x_0)}_{\text{source}}$$

- Probability current $\neq 0$

Example: Reset Brownian motion

[Evans, Majumdar PRL 2011, JPA 2011]

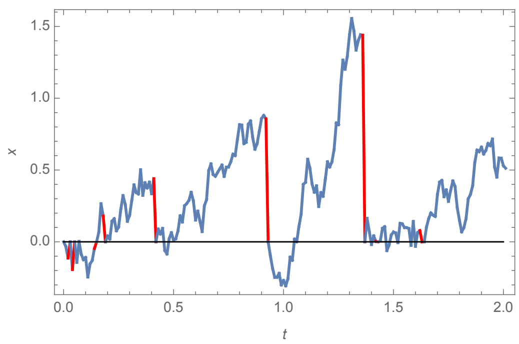
- Reset diffusion equation:

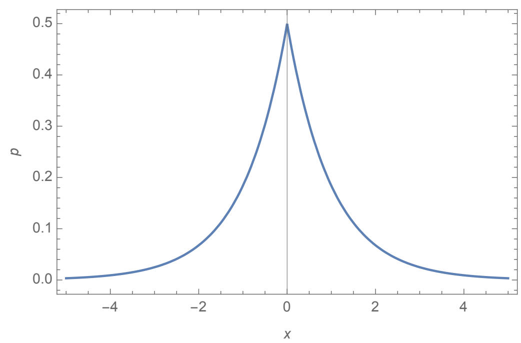
$$\frac{\partial}{\partial t} p(x,t) = D \frac{\partial^2}{\partial x^2} p(x,t) - \Gamma p(x,t) + \Gamma \delta(x-x_0)$$

- Stationary solution:

$$p_{ss}(x) = \frac{\alpha}{2} e^{-\alpha|x-x_0|}, \qquad \alpha = \sqrt{\Gamma/D}$$

- Reset = weak confining force
- No stationary distribution w/o reset





Project 1: Spectral properties

Markov evolutions

$$\partial_t p(x,t) = \mathcal{L} p(x,t)$$

$$\mathcal{L} = -\frac{d}{dx}F + \frac{\sigma^2}{2}\frac{d^2}{dx^2}$$

- Non-Hermitian generator
- Spectrum not real
- $\lambda_{\max} = 0$
- $\mathcal{L} p_{ss} = 0$ stationary density
- Decay dynamics:

$$p(x,t) = p_{ss}(x) + \sum_{i=2}^{\infty} e^{\lambda_i t} f_i(x)$$

Quantum evolutions

$$i\hbar\partial_t \psi(x,t) = \mathcal{H}\psi(x,t)$$

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$$

- Hermitian generator
- Real spectrum
- $E_{\min}$ = ground state
- $\psi_{\min}(x)$
- Rotation dynamics:

$$\psi(x,t) = \sum_{i=1}^{\infty} e^{-i\hbar E_i}\phi_i(x,t)$$

Spectrum of reset processes

$$\partial_t |P(t)\rangle = \mathcal{L}|P(t)\rangle, \qquad |P(t)\rangle = \sum_C P(C,t)|C\rangle, \quad \langle C|C'\rangle = \delta_{CC'}$$

Without reset

- Generator:

$$\mathcal{L} = \sum_{C,C'\neq C} W(C\to C')|C'\rangle\langle C| - \sum_C R(C)|C\rangle\langle C|$$

- Spectral elements:

$$\mathcal{L}|r_i\rangle = \lambda_i|r_i\rangle, \qquad \langle l_i|\mathcal{L} = \lambda_i\langle l_i|$$

- Eigenvalues: $\underbrace{\lambda_1}_{=0} > \lambda_2 \geq \lambda_3 \geq \cdots$
- Dominant eigenvectors:

$$|r_1\rangle = |P_{ss}\rangle, \qquad \langle l_1| = \sum_C \langle C| \equiv \langle -|$$

Spectrum of reset processes (cont'd)

[Rose, HT, Lesanovsky, Garrahan PRE 2018]

With reset

- Generator:

$$\mathcal{L}^\Gamma = \mathcal{L} + \underbrace{\Gamma|C_0\rangle\langle -|}_{\text{reset}} - \underbrace{\Gamma I}_{\text{norm}}$$

- Eigenvalues:

$$\lambda_1^\Gamma = 0, \qquad \lambda_i^\Gamma = \lambda_i - \Gamma \qquad \text{(more dissipation)}$$

- Right eigenvectors: $|r_i^\Gamma\rangle = |r_i\rangle$, $i > 1$
- Left eigenvectors:

$$\langle l_1^\Gamma| = \langle -|, \qquad \langle l_i^\Gamma| = \langle l_i| + \frac{\Gamma\langle l_i|C_0\rangle}{\lambda_i - \Gamma}\langle -|$$

- Stationary distribution:

$$|P_{\text{ss}}^\Gamma\rangle = |P_{\text{ss}}\rangle - \sum_{i=2}^{D} \frac{\Gamma\langle l_i|C_0\rangle}{\lambda_i - \Gamma}|r_i\rangle$$

Example: Reset Brownian motion

$$\mathcal{L}p(x) = D\frac{d^2p(x)}{dx^2}, \qquad x \in [-L/2, L/2) \qquad \text{periodic b.c.}$$

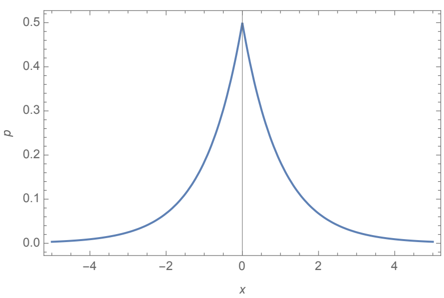
- Free quantum particle in box
- Spectral elements:

$$\lambda_n = -D\left(\frac{2\pi n}{L}\right)^2, \quad n \in \mathbb{Z}, \qquad r_n(x) = l_n(x) = \frac{1}{\sqrt{L}}e^{2\pi i n x/L}$$

- Stationary distribution:

$$p_{\text{ss}}^\Gamma(x) = \frac{\Gamma}{L}\sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n x/L}}{D(\frac{2\pi n}{L})^2 + \Gamma}$$

$$\stackrel{L\to\infty}{=} \Gamma\int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{e^{ikx}}{Dk^2 + \Gamma} = \frac{\alpha}{2}e^{-\alpha|x|}$$



Open quantum systems

- Master equation:

$$\partial_t \rho = \mathcal{L}(\rho)$$

- Lindblad operator:

$$\mathcal{L}(\rho) = \underbrace{-i[H,\rho]}_{\text{closed evolution}} + \underbrace{\sum_j \left[J_j \rho J_j^\dagger - \frac{1}{2}\{J_j^\dagger J_j, \rho\}\right]}_{\text{system-environment interaction}}$$

- Spectral elements:

$$\mathcal{L}(R_i) = \lambda_i R_i, \qquad \mathcal{L}^\dagger(L_i) = \lambda_i^* L_i \qquad \text{(eigen-matrices)}$$

Reset dynamics

- Reset state: $|\psi\rangle$
- Reset operators: $J_i^\Gamma = \sqrt{\Gamma}|\psi\rangle\langle\phi_i|$
- Generator:

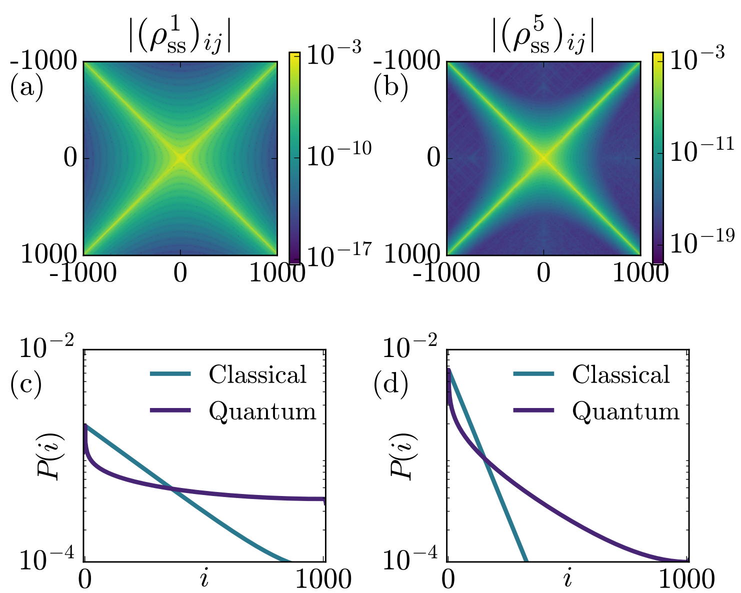
$$\mathcal{L}_\Gamma(\rho) = \mathcal{L}(\rho) + \Gamma\, \mathrm{Tr}(\rho)|\psi\rangle\langle\psi| - \Gamma\rho$$

Example: Quantum random walk

- Coherent hopping on a chain
- No environment (closed system) + reset
- Hamiltonian:

$$H = \gamma \sum_{i=1}^{L-1} \big(|x+1\rangle\langle x| + |x\rangle\langle x+1|\big) \qquad \text{periodic b.c.}$$

- Reset: $J_i^\Gamma = \sqrt{\Gamma}|0\rangle\langle\phi_i|$
- Not same as projective measurement
- Localization with reset
- Compared with classical random walk
- $\Gamma = 1, 5$
- $N = 2001$ sites



Project 2: Large deviations

[HT Physics Reports 2009]

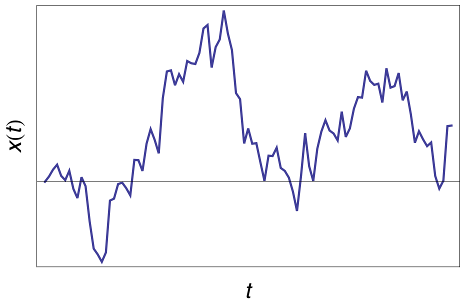
- Markov process: X_t
- Observable:

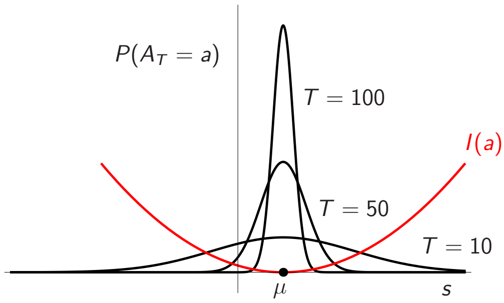
$$A_T = \frac{1}{T}\int_0^T f(X_t)\,dt$$

- Large deviation approximation:

$$P(A_T = a) \approx e^{-T I(a)}, \quad T \to \infty$$

- Rate function: $I(a)$





Physical observables

- Nonequilibrium: Work, heat, entropy production
- Equilibrium: Rate function = entropy

Spectral calculation

[HT Physica A 2018]

Scaled cumulant function

$$\Lambda(k) = \lim_{T\to\infty} \frac{1}{T} \ln E[e^{TkA_T}]$$

- $k \in \mathbb{R}$

Gärtner-Ellis Theorem

$\Lambda(k)$ differentiable, then

1. $P(A_T = a) \approx e^{-TI(a)}$
2. $I(a) = \sup_k \{ka - \Lambda(k)\}$

Markov processes

$$\mathcal{L}_k r_k = \Lambda(k) r_k$$

- Tilted (twisted) operator: $\mathcal{L}_k = \mathcal{L} + kf$
- Dominant eigenvalue: $\Lambda(k)$
- Dominant eigenfunction: r_k
- Perron-Frobenius-type result
- Rate function = Legendre transform of SCGF

Generating function with reset

[Meylahn, Sabhapandit, HT PRE 2015]

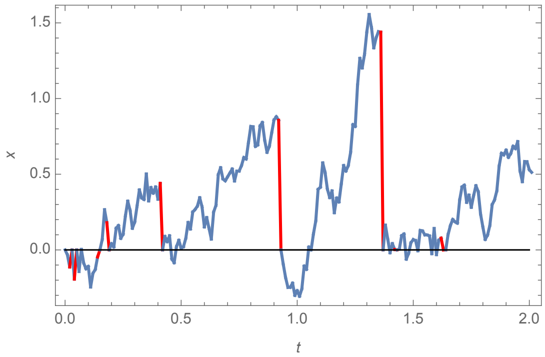
- Generating function:

$$G_\Gamma(x,k,T) = E_x[e^{TkA_T}] = E[e^{k\int_0^T f(X_t)dt} | X_0 = x]$$

- Renewal representation:

$$\underbrace{G_\Gamma(x,k,T)}_{\text{with reset}} = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \underbrace{G_0(x_r,k,\tau_i)}_{\text{no reset}} \; P(n \text{ resets})\; \delta\Big(\textstyle\sum_i \tau_i - T\Big)$$

- n resets
- Reset times: $\tau_1, \ldots, \tau_n$
- $G_\Gamma = G_0$ between resets
- $\sum_i \tau_i = T$



Generating function with reset (cont'd)

- Laplace transform:

$$\tilde{G}_\Gamma(x,k,s) = \int_0^\infty G_\Gamma(x,k,T)\, e^{-sT}\, dT$$

- Renewal representation:

$$\tilde{G}_\Gamma(x,k,s) = \tilde{G}_0(x,k,s+\Gamma) \sum_{n=0}^{\infty} \Gamma^n\, \tilde{G}_0(x_r,k,s+\Gamma)^n$$

$$\tilde{G}_\Gamma(x,k,s) = \frac{\tilde{G}_0(x,k,s+\Gamma)}{1-\Gamma\tilde{G}_0(x_r,k,s+\Gamma)}$$

Long-time behavior

$$G_\Gamma(x,k,T) \sim e^{T\Lambda_\Gamma(k)} \quad \Longleftrightarrow \quad \tilde{G}_\Gamma(x,k,s) \sim \frac{1}{s-\Lambda_\Gamma(k)}$$

- SCGF = largest pole of $\tilde{G}_\Gamma$
- $\tilde{G}_\Gamma$ = transform of no-reset $\tilde{G}_0$

Example: Ornstein–Uhlenbeck process with reset

- Process:

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

- Observable:

$$A_T = \frac{1}{T}\int_0^T X_t dt$$

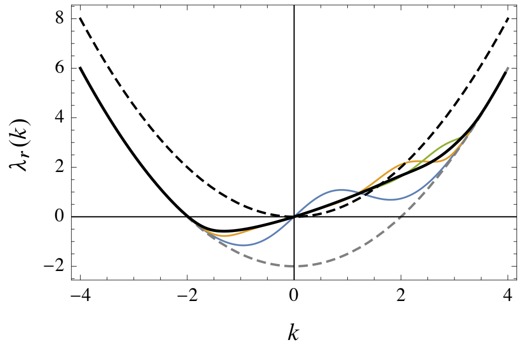
- Spectral decomposition:

$$G_0(x,k,T) = \sum_{i=0}^{\infty} \psi_{k,i}(x) e^{\Lambda_{0,i}(k)T}$$

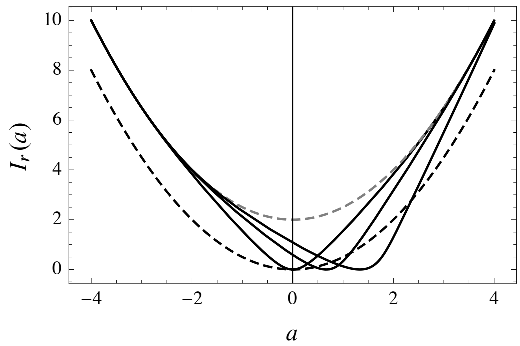
- Asymptotics:

$$\Lambda_\Gamma(k) \approx \Lambda_0(k) - \Gamma$$

$$I_\Gamma(a) \approx I_0(a) + \Gamma$$



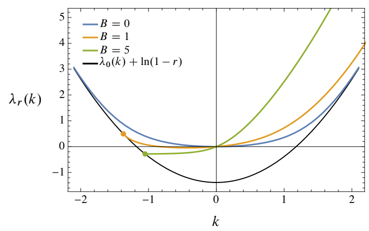
Dash: $\Gamma = 0$
Black line: $\Gamma = 1$
Grey dash: Asymptotics



Other results

Dynamical phase transitions [Harris, HT JPA 2017]

- Phase transitions in fluctuations
- Singularities in $\Lambda_\Gamma(k)$ and $I_\Gamma(a)$
- Mapping to DNA models with phase transitions

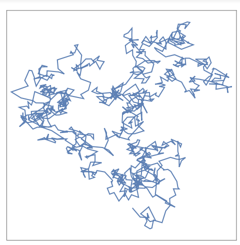


Variants

- Spatially dependent reset rate $\Gamma(x)$
- Non-exponential reset times (non-Poisson = non-Markov)
- Non-instantaneous return, waiting at reset state, etc.

Random search

- Reset search more efficient
- Mean time to random target
- Mean first passage time reduced with reset



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