Nonconcave entropies and multifractal spectra

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Outline

1. Large deviation theory
   - Nonconvex rate functions
   - Nonconcave entropies
2. Nonconcave multifractal spectra
3. New method for nonconcave spectra

H. Touchette and C. Beck
Nonconcave entropies in multifractals
and the thermodynamic formalism
J. Stat. Phys. 125, 455, 2006, cond-mat/0507379

H.Touchette
Methods for calculating nonconcave entropies
J. Stat. Mech. P05008, 2010, arxiv:1003.0382

Large deviation theory

- Random variable: A_n
- Probability density: $P(A_n = a)$

Large deviation principle (LDP)

$$P(A_n = a) \approx e^{-nI(a)}$$

- Meaning of $\approx$:

$$\ln P(a) = -nI(a) + o(n)$$
$$\lim_{n\to\infty} -\frac{1}{n} \ln P(a) = I(a)$$

- Rate function: $I(a) \geq 0$

Goals of large deviation theory

1. Prove that a large deviation principle exists
2. Calculate the rate function

Result 1: Varadhan's Theorem

- LDP:

$$P(A_n = a) \approx e^{-nI(a)}$$

- Exponential expectation:

$$E[e^{nf(A_n)}] = \int e^{nf(a)}\, P(A_n = a)\, da$$

- Limit functional:

$$\lambda(f) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nf(A_n)}]$$

- Courant Institute
- Abel Prize 2007

Theorem: Varadhan (1966)

$$\lambda(f) = \max_a \{f(a) - I(a)\}$$

Special case: $f(a) = ka$

$$\lambda(k) = \max_a \{ka - I(a)\}$$

Result 2: Gärtner-Ellis Theorem

Scaled cumulant generating function (SCGF)

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkA_n}], \qquad k \in \mathbb{R}$$

Theorem: Gärtner (1977), Ellis (1984)

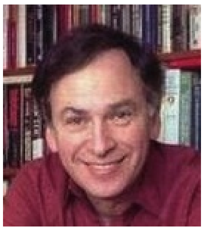
If $\lambda(k)$ is differentiable, then

1. Existence of LDP:

$$P(A_n = a) \approx e^{-nI(a)}$$

2. Rate function:

$$I(a) = \max_k \{ka - \lambda(k)\}$$



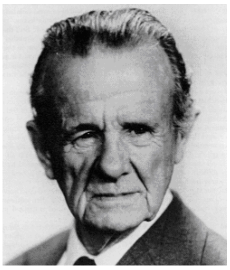
Richard S. Ellis

J. Gärtner

- $I(a)$ is the Legendre transform of $\lambda(k)$
- $I(a)$ is strictly convex

Sums of IID random variables

Cramér (1938)



- Sample mean:

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad X_i \sim p(x), \quad IID$$

- SCGF:

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkS_n}] = \lim_{n\to\infty} \frac{1}{n} \ln E\left[\prod_{i=1}^{n} e^{kX_i}\right] = \ln E[e^{kX}]$$

Gaussian

$$\lambda(k) = \mu k + \frac{\sigma^2}{2}k^2, \quad k \in \mathbb{R}$$

$$I(s) = \frac{(s-\mu)^2}{2\sigma^2}, \quad s \in \mathbb{R}$$

Exponential

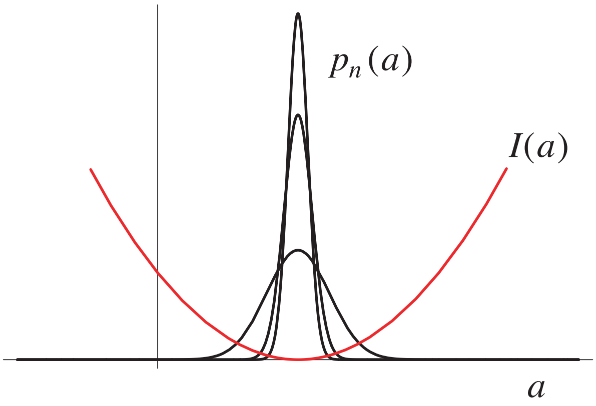
$$\lambda(k) = -\ln(1-\mu k), \quad k < \frac{1}{\mu}$$

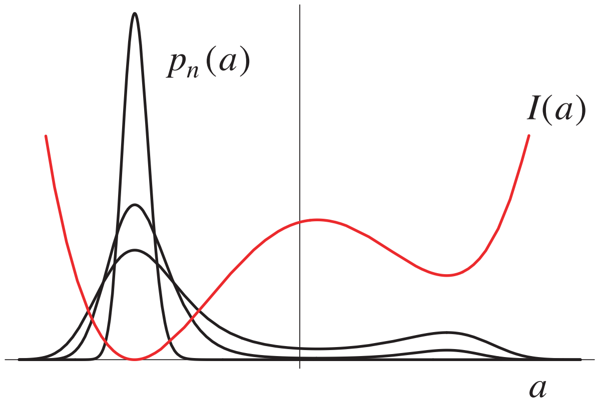
$$I(s) = \frac{s}{\mu} - 1 - \ln\frac{s}{\mu}, \quad s > 0$$

General properties

$$P(A_n = a) \approx e^{-nI(a)}$$

- Most probable value = typical value = min and zero of I
- Zero of I = Law of Large Numbers
- Local parabolic minimum = Central Limit Theorem





Nonconvex rate function

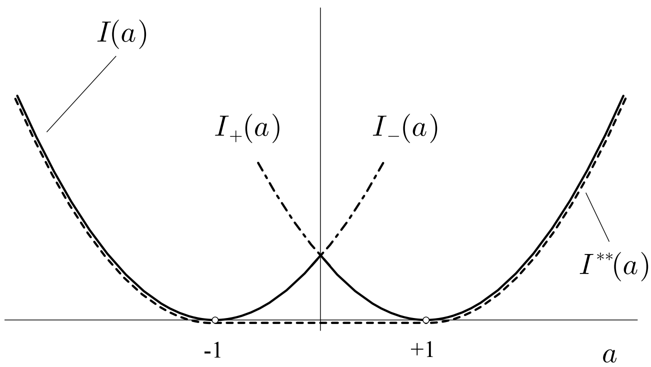
Ioffe (1993)

- Mixed Gaussian sample mean:

$$A_n = Y + \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad X_i \sim \mathcal{N}(0,1), \quad Y \sim \mathcal{U}\{-1,1\}$$

- LDP: $P(A_n = a) \approx e^{-nI(a)}$
- Rate function:

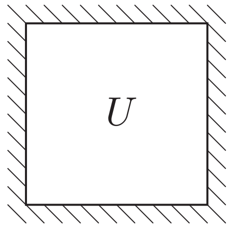
$$I(a) = \begin{cases} \frac{(a+1)^2}{2} & a \leq 0 \\ \frac{(a-1)^2}{2} & a > 0 \end{cases}$$



- $I(a)$ cannot be obtained from GE Theorem
- $\lambda(k)$ has nondifferentiable point
- Legendre transform of $\lambda(k)$ gives convex envelope $I^{**}(a)$

Nonconcave entropies in statistical mechanics

- N particles
- Microscopic configuration: $\omega = x_1, x_2, \ldots, x_N$
- Energy: $U(\omega)$



Entropy

- Density of states:

$$\rho(u) = |\{\omega : U/N = u\}|$$

- LDP: $\rho(u) \approx e^{Ns(u)}$
- Entropy:

$$s(u) = \lim_{N\to\infty} \frac{1}{N} \ln \rho(u)$$

Free energy

- Partition function:

$$Z(\beta) = \sum_{\text{states}} e^{-\beta U}$$

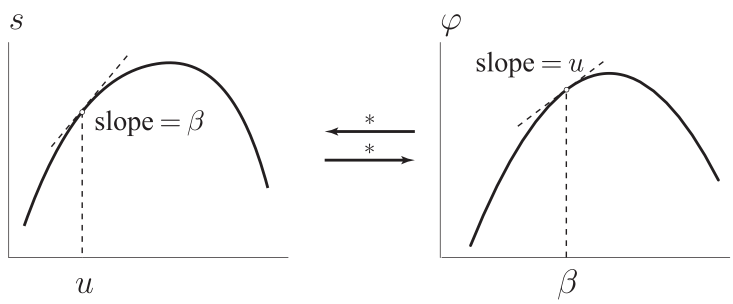
- LDP: $Z(\beta) \approx e^{-N\varphi(\beta)}$
- Free energy:

$$\varphi(\beta) = \lim_{N\to\infty} -\frac{1}{N} \ln Z(\beta)$$

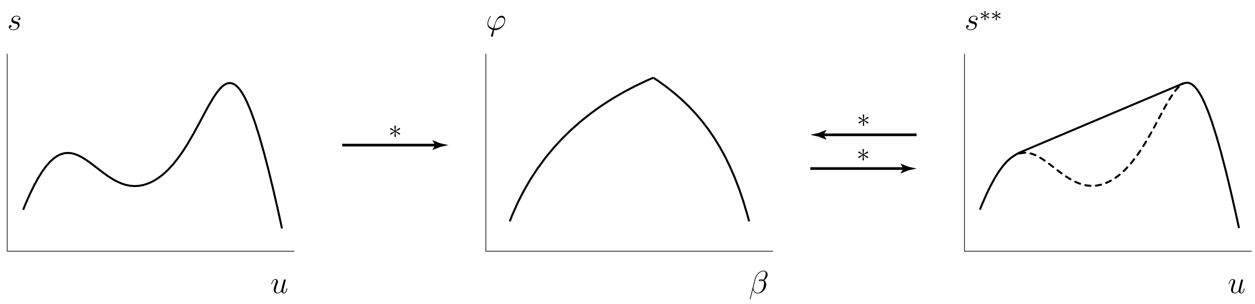
Concave vs nonconcave entropy

- Concave entropy:

$$\varphi(\beta) = \inf_u \{\beta u - s(u)\}$$
$$s(u) = \inf_\beta \{\beta u - \varphi(\beta)\}$$



- Nonconcave entropy:

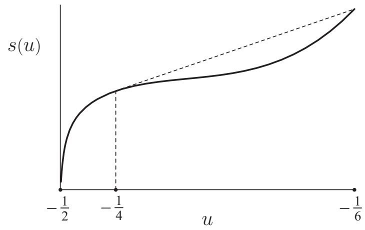


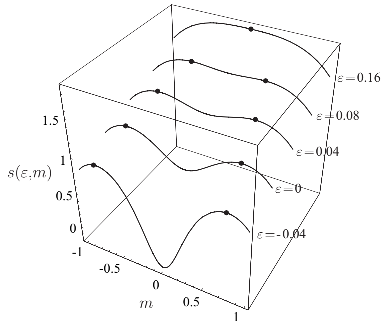
- $s^{**}(u) =$ concave envelope of $s(u)$
- $\varphi(\beta)$ is nondifferentiable for $s(u)$ nonconcave

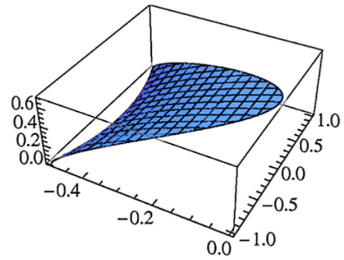
Examples

(Campa, Dauxois & Ruffo Phys Rep 2009)

- Gravitational systems
  - Stars
  - Globular clusters
- Spin systems
  - Potts model
  - ϕ^4 model
- 2D turbulence
- Optical lattices
- Quantum spins
  - Heisenberg model





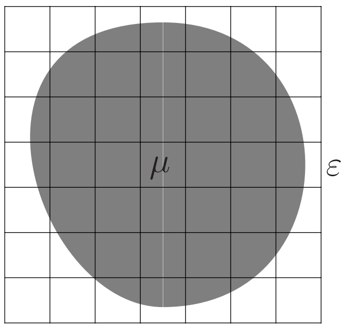


- Long-range interaction
- Short-range interaction $\Rightarrow$ concave entropy

Multifractal formalism

- Measure: $\mu(x)$
- Coarse-graining: $p_{i,\varepsilon} = \int_{i\text{th box}} d\mu(x)$
- Local exponent:

$$p_{i,\varepsilon} \sim \varepsilon^{\alpha_i}, \qquad \alpha_{i,\varepsilon} = \frac{\ln p_{i,\varepsilon}}{\ln \varepsilon}$$



Structure function

- Partition function:

$$S_\varepsilon(q) = \sum_i \varepsilon^{q\alpha_{i,\varepsilon}}$$

- LDP: $S_\varepsilon(q) \sim \varepsilon^{\tau(q)}$
- Structure function:

$$\tau(q) = \lim_{\varepsilon\to 0} \frac{\ln S_\varepsilon(q)}{\ln \varepsilon}$$

Distribution of local exponents

- Histogram:

$$n_\varepsilon(\alpha) = \#\text{ boxes with } \alpha_\varepsilon \in [\alpha, \alpha + d\alpha]$$

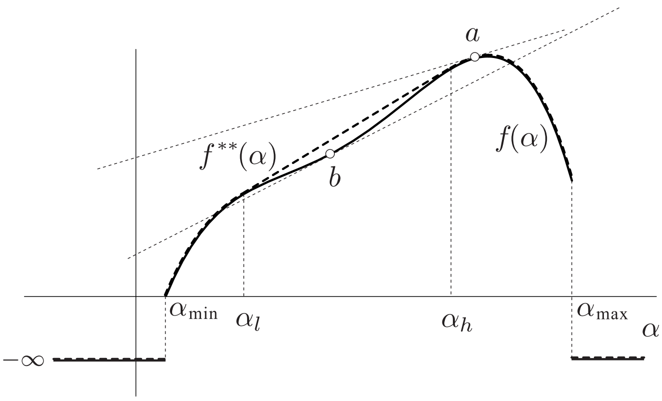
- LDP: $n_\varepsilon(\alpha) \sim \varepsilon^{-f(\alpha)}$
- Multifractal spectrum:

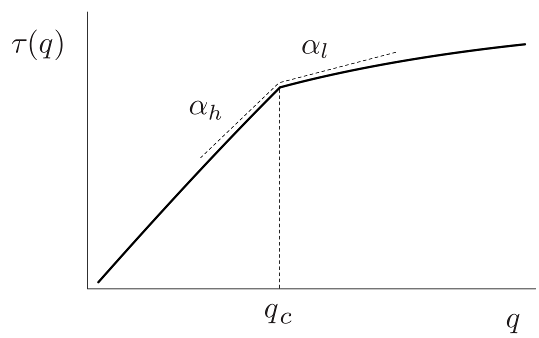
$$f(\alpha) = \lim_{\varepsilon\to 0} -\frac{\ln n_\varepsilon(\alpha)}{\ln \varepsilon}$$

Concave vs nonconcave spectrum

(HT & Beck JSP 2005)

- Varadhan:
$$\tau(q) = \inf_{\alpha}\{q\alpha - f(\alpha)\}$$
- Concave spectrum:
$$f(\alpha) = \inf_{q}\{q\alpha - \tau(q)\}$$
- Nonconcave spectrum:
$$f(\alpha) \leq f^{**}(\alpha) = \inf_{q}\{q\alpha - \tau(q)\}$$

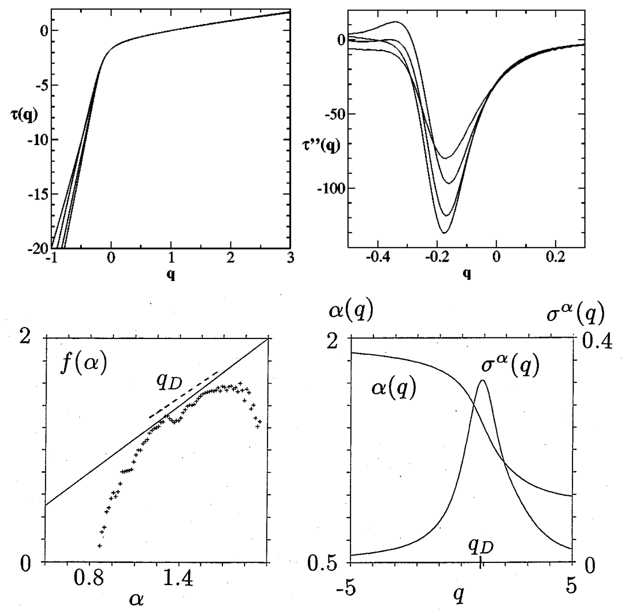




Examples from physics

(HT & Beck JSP 2005)

- Turbulence
  - $dv(l) = |v(x+l) - v(x)|$
  - $\langle (dv)^p \rangle \sim l^{\zeta_p}$
- Limited diffusion
  - Jensen *et al.* PRE 2002
- Chaotic systems
  - Strange attractors
  - Hénon map
  - Driven damped pendulum
  - Tominaga *et al.* PTP 1990
- Dynamical indices spectra



- Concavity of $f(\alpha)$ assumed in most cases
- Related to multifractal or q phase transitions
- How to obtain nonconcave $f(\alpha)$?

Generalized canonical ensembles

(Costeniuc, Ellis, HT & Turkington, JSP 2005; PRE 2006)
(HT & Beck JSP 2005)

Canonical

$$S_\varepsilon(q) = \sum_i \varepsilon^{q\alpha_{i,\varepsilon}}$$

$$\tau(q) = \lim_{\varepsilon\to 0} \frac{\ln S_\varepsilon(q)}{\ln \varepsilon}$$

Generalized canonical

$$S_{g,\varepsilon}(q) = \sum_i \varepsilon^{q\alpha_{i,\varepsilon}+g(\alpha_{i,\varepsilon})}$$

$$\tau_g(q) = \lim_{\varepsilon\to 0} \frac{\ln S_{g,\varepsilon}(q)}{\ln \varepsilon}$$

Generalized Gärtner-Ellis Theorem

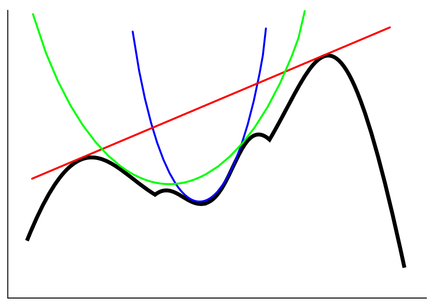
- Choose g
- If $\tau_g(q)$ is differentiable, then

$$f(\alpha) = \inf_q \{q\alpha - \tau_g(q)\} + g(\alpha)$$

Choice for g

$$S_{g,\varepsilon}(q) = \sum_i \varepsilon^{q\alpha_{i,\varepsilon}+g(\alpha_{i,\varepsilon})}$$

- Canonical:
  - $g = 0$
  - $g = \text{const}$
  - $g = \gamma\alpha$
- Gaussian: $g(\alpha) = \gamma\alpha^2$
- Betrag: $g(\alpha) = \gamma|\alpha|$
- Others?



Universal equivalence

- Any spectrum can be obtained with Gaussian ensemble
- $\gamma > \gamma_c = \max f''(\alpha)$
- γ related to local curvature of $f(\alpha)$
- Supporting parabola interpretation
- Also works for rate functions / entropies

Example: Mixed Gaussian sample mean

- Gaussian SCGF:

$$\lambda_\gamma(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkA_n + n\frac{\gamma}{2}A_n^2}]$$

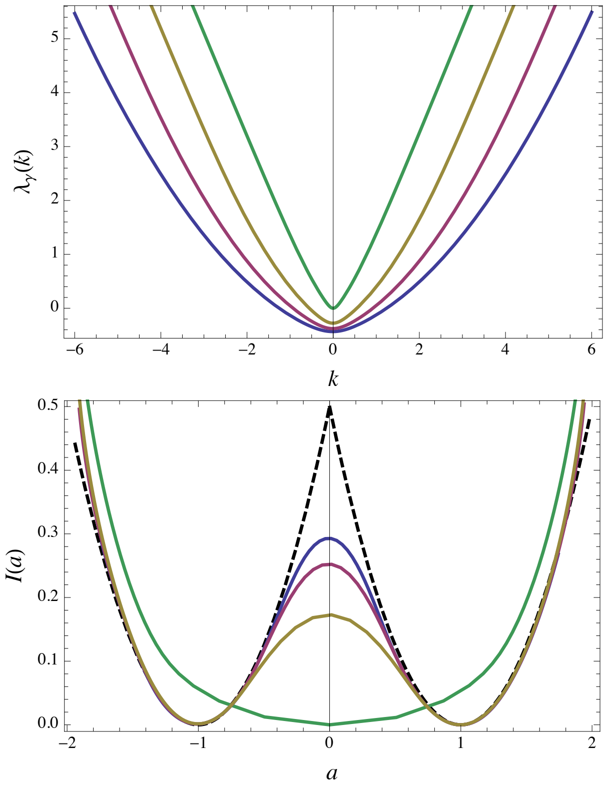
- Sample: $\{A_n^{(j)}\}_{j=1}^L$
- SCGF estimator:

$$\lambda_\gamma(k) = \frac{1}{n} \ln \frac{1}{L} \sum_{j=1}^{L} e^{nkA_n^{(j)} + n\frac{\gamma}{2}(A_n^{(j)})^2}$$

- Rate function:

$$I(a) = \inf_k \{ka - \lambda_\gamma(k)\} + \frac{\gamma}{2}a^2$$

- Simulations: $L = 2000$, $n = 10$
  $\gamma = -3, -2, -1, 0$



Open questions

Generalized formalism

- Apply to real multifractals / signals
- Numerical / sampling issues for Gaussian ensemble
- Other functions g?

Previous studies

- Revisit past studies of (concave?) spectra
- Nonconcave for $\varepsilon > 0$ but concave for $\varepsilon \to 0$?

Other

- Source of nonconcavity for multifractals
- Physics: Long-range interaction or mixed phases
- Long-range time correlations?

References

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