Doing maths with computers

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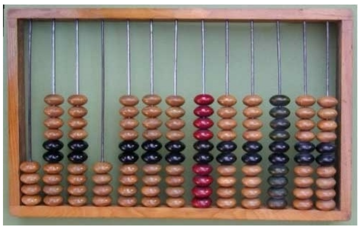
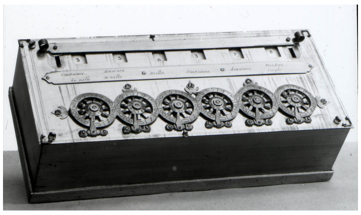
Presentation available at
http://www.maths.qmul.ac.uk/~ht

Outline

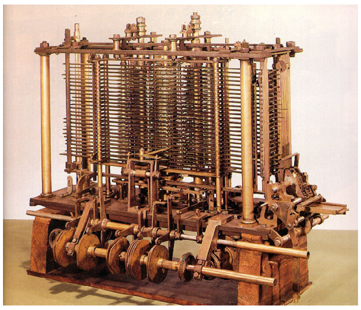
Early mechanical calculators

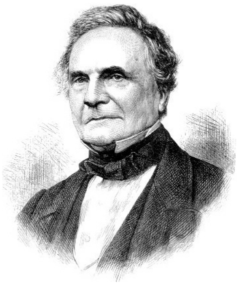


- Abacus
- Mesopotamia (2500 BC), Roman Empire, China, etc.
- Quick arithmetic operations



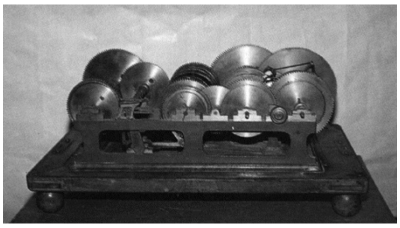
- Pascal's calculator 1645 (Pascalina or the Arithmetique)
- Blaise Pascal (1623-1662)
- Add and substract numbers





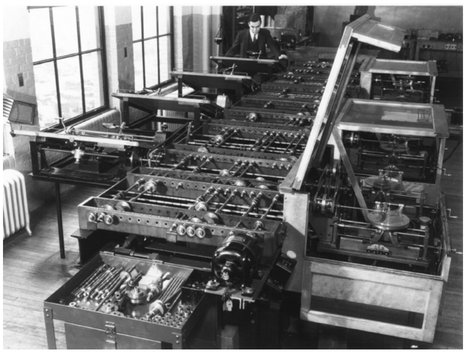
- Difference engine 1822
- Charles Babbage (1791-1871)
- Compute values of polynomial functions

More mechanical computers



- Leonardo Torres Quevedo (1852-1936)
- Computes real and complex roots of trinomial equations $x^p + x + r = 0$

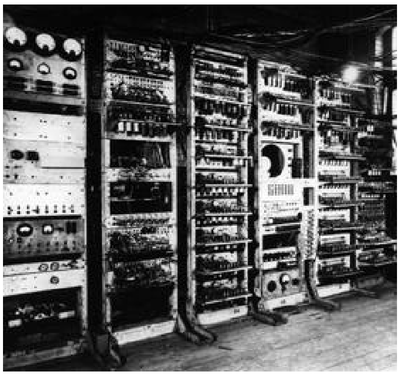
- Isograph, AT&T, 1937
- Calculates roots of polynomials up to degree 15



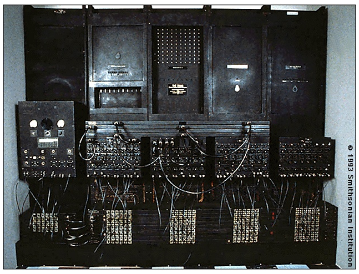
- Differential analyzer, 1910s-1920s
- Solves differential equations

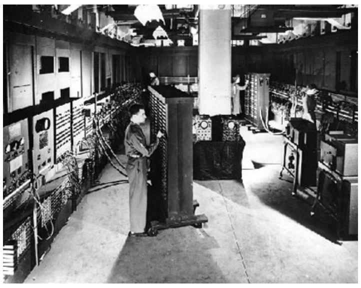
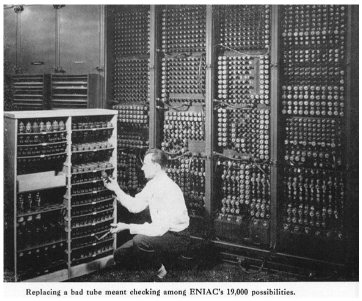
First universal computers

1945: ACE (Automatic Computing Machine)
Designed by Alan Turing


1946: ENIAC (Electronic Numerical Integrator And Computer)
Designed by John von Neumann



Modern computers

Personal computers / calculators



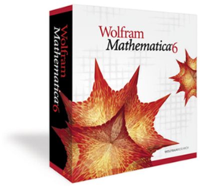
Supercomputers


Fastest computer

- DOE Roadrunner, USA, 1400 TeraFlops $= 1400 \times 10^{12}$ ops/sec
- List at www.top500.org

Mathematical softwares

Maple (commercial)
Maplesoft (Canada)
maplesoft.com



Mathematica (commercial)
Wolfram Research (USA)
wolfram.com

Sage (open source, free)
sagemath.org

Magma (non-commercial, license fee)
University of Sydney
magma.maths.usyd.edu.au/magma/

- Numerical computations
- Symbolic computations

Representing large numbers

Question 1

What's the biggest number you can write on a typical pocket calculator?

Solution:

Question 2

What's the biggest number that you think a computer can represent?

Solution:

Question 3

What limits us in representating numbers on computers?

Solution:

Crunching big numbers

Question 4

Write down the value of 4!

Solution:

Question 5

How many digits do you think 1000! has?

Solution:

Question 6

Write the largest number you can think of using 4 digits arranged in any way you like.

Solution:

Solving simple polynomial equations

- Some equations can be solved by hand
- No need to use computers

Question 7

Write down the solution of $2x + 4 = 8$.

Solution:

Question 8

Write down the solutions of the general equation

$$ax^2 + bx + c = 0$$

where a, b, and c are constants. How many solutions are there?

Solution:

Cubic equations

- Some equations are more difficult to solve
- Computers begin to be useful

Question 9

Find all the solutions of $x^3 - x = 0$.

Solution:

Question 10

Consider the general equation

$$ax^3 + bx^2 + cx + d = 0$$

where a, b, c, and d are constants. Do you know the formula for the solutions of this equation? How many solutions are there?

From cubic to quintic equations

Here's one solution of the equation of degree 3 of the previous page:

$$x = -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} +$$
$$\frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{32^{1/3}a}$$

Question 11

Consider the general equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

of degree 4. Is there an explicit formula for the solutions of this equation?

Solution:

Mathematical fact

Solutions of equations of degree 5 (quintics) cannot be expressed in terms of the four arithmetic operations and roots only.

Transcendental equations

- Some equations can't be solved in closed form
- But they can be solved numerically on a computer

Question 12

Consider the equation

$$e^{-x} = x.$$

Can you solve it? That is, can you find the numerical value that verifies this equation?

Solution:

Transcendental equations

Question 13

Find the solution of

$$\cos(x) = x$$

in the interval $[0, \pi]$.

Solution:

Question 14

Find the two solutions of the equation

$$\cos(x^3) - x^2 = 0.$$

Solution:

Plotting functions of one variable

Question 15

Can you plot the following functions?

a) $f(x) = x^2$

b) $f(x) = \sin(x)$

c) $f(x) = \sin(\frac{1}{x})$

Solution:

Plotting functions of two variables

Question 16

Try to plot the following functions:

a) $f(x, y) = x^2 + y^2$

b) $f(x, y) = x^2 - y^2$

c) $f(x, y) = x \sin(x) \cos(y)$

Solution:

Plotting function of three variables

- Can we plot a function $f(x, y, z)$ of three variables?

Question 17

The equation of a sphere of radius r is

$$x^2 + y^2 + z^2 = r^2$$

Try to plot the sphere of radius 2 in 3D.

Solution:

The prime numbers

Question 18

What's a prime number?

Solution:

Question 19

Write down the first 10 primes.

Solution:

Question 20

Is 7918 a prime? What about 7919?

Solution:

Distribution of primes

Here are the primes between 2 and 100:

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

Question 21

Do you see any order in the way the primes appear?

Solution:

Question 22

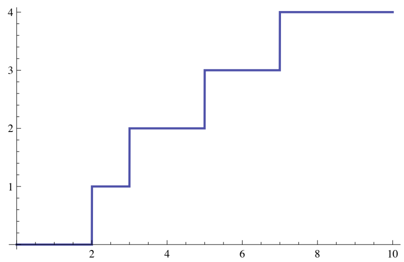
Is there a maximum prime number? In other words, is there a finite or an infinite number of prime numbers?

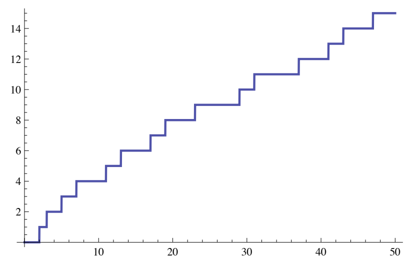
Solution:

Plotting the distribution of primes

- Cumulative distribution:

$$N(x) = \text{Number of primes smaller or equal to } x$$





Question 23

Do you see any pattern in the way $N(x)$ grows?

Solution:

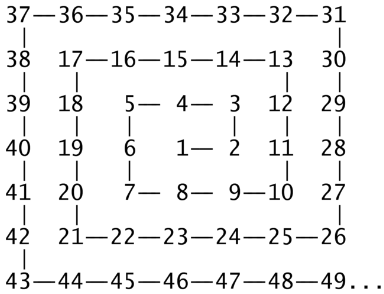
Question 24

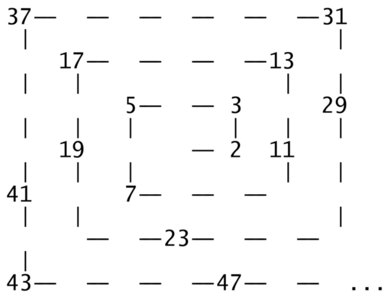
How quickly does $N(x)$ grow?

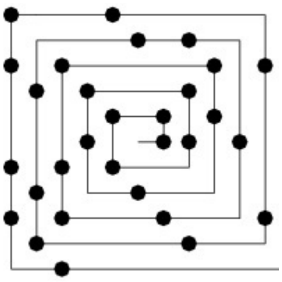
Solution:

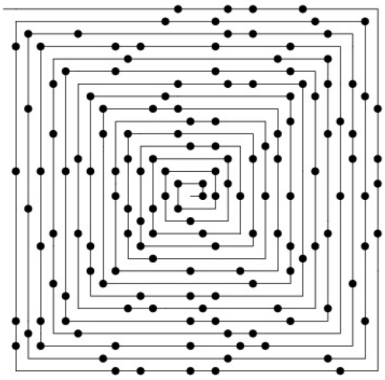
Order in the primes: The Ulam prime spiral

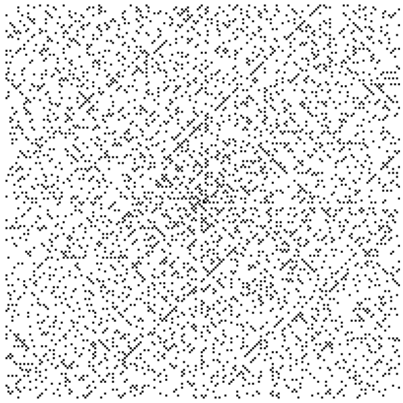
Stanislaw Ulam, 1963



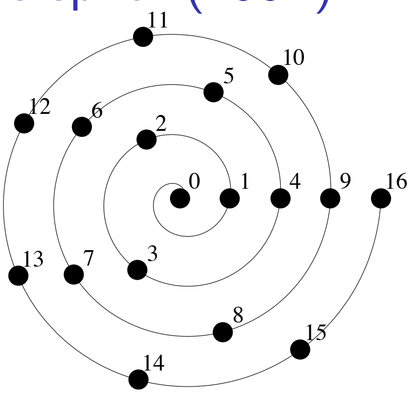




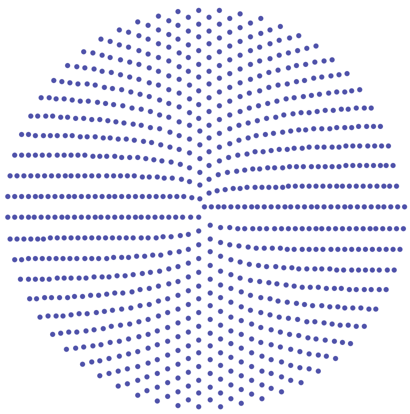


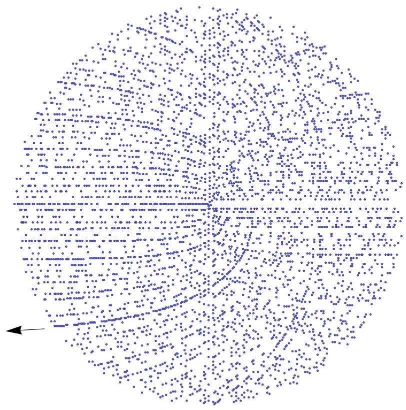


Sacks's spiral (1994)



Archimedean spiral





$x^2 + x + 41$ (Euler, 1772)

What's programming?

- Instructing a computer to perform a task
- Writing down the recipe for a computation
- Computer's language = **programming language**

Programming language

Java, C, C++, Pascal, Fortran, Basic,...

Basis of all languages

- Arithmetic operations: $+$, $-$, $\times$, $\div$
- Functions: e.g. $\sin(x)$
- Repeated execution (loops)
- Conditional execution (ifs)

A simple example

Problem

Add all the prime numbers between 1 and 100

Recipe

$sumvalue = 0$
1 prime? no
2 prime? yes $\Rightarrow sumvalue + 2$
3 prime? yes $\Rightarrow sumvalue + 3$
4 prime? no
$\vdots$
99 prime? no
100 prime? no
Print final *sumvalue*

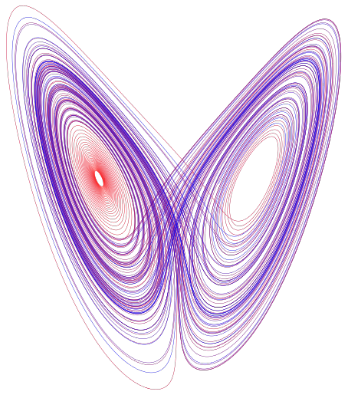
Program

```
sumvalue:=0;
for i from 1 to 100 do
   if isprime(i)=true then
      sumvalue:=sumvalue+i;
   end if;
end do;
sumvalue;
```

Another example: The Lorenz attractor

Edward Lorenz, 1963

- Position: $(x(t), y(t), z(t))$
- Equations of motion:



$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Basic code

```
for i from 0 to t/dt do
  x:=x+sigma*(y-x)*dt;
  y:=y+(rho*x-x*z-y)*dt;
  z:=z+(x*y-beta*z)*dt;
end do:
```


Extra: Program for cumulative distribution of primes

- Cumulative distribution:

$$N(x) = \text{Number of primes smaller or equal to } x$$

Recipe

1. Scan all number from 1 to x
2. Add 1 to a counter when a prime is encountered

Algorithm

$nprimes = 0$
1 prime? no
2 prime? yes $\Rightarrow nprimes + 1$
3 prime? yes $\Rightarrow nprimes + 1$
4 prime? no
$\vdots$
x prime?
Print final *nprimes*

Program

```
nprimes:=0;
for i from 1 to x do
  if isprime(i)=true then
    nprimes:=nprimes+1;
  end if;
end do;
nprimes;
```

If you want to know more...

- History of computers:
  - Wikipedia
- Top 500 computers in the world:
  - http://www.top500.org
- Solving quintic equations:
  - http://library.wolfram.com/examples/quintic/
- Prime numbers:
  - Wikipedia
  - Marcus du Sautoy, The Music of the Primes, Harper Collins, 2003
- Prime spiral:
  - Wikipedia
  - http://www.numberspiral.com/
  - http://mathworld.wolfram.com/PrimeSpiral.html
- Lorenz attractor:
  - Wikipedia
  - http://mathworld.wolfram.com/LorenzAttractor.html

http://www.maths.qmul.ac.uk/~ht

Notes