# Convex invertible cones and Nevanlinna-Pick interpolation: the suboptimal case

Alma van der Merwe[1]

University of the Witwatersrand

65th SAMS Congress
6-8 December 2022, Stellenbosch University

[1] A joint work with Sanne ter Horst

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### Classical Nevanlinna-Pick interpolation (Pick 1916, Nevanlinna 1929)

Given points $\lambda_1, \dots, \lambda_n$ in the unit disc $\mathbb{D}$ (or right half plane $\Pi_+$) and values $\eta_1, \dots, \eta_n$, find an analytic function $f$ that maps $\mathbb{D}$ into $\overline{\mathbb{D}}$ (resp., $\Pi_+$ into $\overline{\Pi}_+$) so that $f(\lambda_k) = \eta_k$ for $k = 1, \dots, n$. There exists a solution iff the Pick matrix $\mathbb{P}$ is positive semidefinite, where $\mathbb{P} := \left[\frac{1-\overline{\eta}_i\eta_j}{1-\overline{\lambda}_i\lambda_j}\right]_{i,j=1}^n$ for the unit disc case and $\mathbb{P} := \left[\frac{\overline{\eta}_i+\eta_j}{\overline{\lambda}_i+\lambda_j}\right]_{i,j=1}^n$ for the right half plane case.

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Consider the class $\mathcal{PRO}$ of rational complex functions $f$ for which

$$f\left(\Pi_+\right) \subseteq \overline{\Pi}_+ \text{ (positive)}, \ f(\mathbb{R}) \subseteq \mathbb{R} \text{ (real)}, \ -f(z) = \overline{f\left(-\overline{z}\right)} \ (odd).$$

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### The Cohen-Lewkowicz Nevanlinna-Pick problem

Given $A, B \in \mathbb{R}^{n\times n}$ with $A$ Lyapunov regular, when is there a $f \in \mathcal{PRO}$ s.t. $f(A) = B$? Classical Nevanlinna-Pick interpolation occurs when $A$ and $B$ are diagonal. Evaluating a (rational) function in a Jordan block $J_k(\lambda)$ corresponds to evaluating $f$ and derivatives up to order $k-1$ in $\lambda$. So, an interpolation condition $f(A) = B$ in matrix point $A$ can combine derivative interpolation conditions at various points.

## Convex invertible cones



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- For $\mathcal{X} \subset \mathbb{F}^{n\times n}$, $\{\mathcal{X}\}'_{\mathbb{F}} := \{Y \in \mathbb{F}^{n\times n} \colon YX = XY,\ X \in \mathcal{X}\}$ is a **cic** in $\mathbb{F}^{n\times n}$.
- For $A \in \mathbb{F}^{n\times n}$ the non-strict and strict Lyapunov solutions of $A$:

$$\overline{\mathcal{H}}(A) := \{H \in \mathcal{H}_n : HA + A^*H \in \overline{\mathcal{P}}_n\},$$
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**Corollary.** *If $A \in \mathbb{F}^{n\times n}$ has no eigenvalues on $\mathrm{i}\mathbb{R}$, then*

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**Question.** How does one determine if $B \in \mathcal{C}(A)$? If $B \in \mathcal{C}(A)$, then $B \in \{A\}''_{\mathbb{F}}$.

# The Lyapunov order and operator

**Definition.** For $A, B \in \mathbb{F}^{n\times n}$ we say $B$ Lyapunov dominates $A$, denoted $A \preceq_{\mathcal{L}} B$, if each (non-strict) Lyapunov solution for $A$ is a Lyapunov solution for $B$, i.e., $\overline{\mathcal{H}}(A) \subset \overline{\mathcal{H}}(B)$.

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So, $B$ Lyapunov dominates $A$ if and only if $\mathcal{L}_{A,B}$ is a positive matrix map.

**Definition.** For positive integers $q$ and $n$, the linear matrix map

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**Definition.** For positive integers $q$ and $n$, the linear matrix map

$$\mathcal{L} : \mathbb{F}^{q \times q} \to \mathbb{F}^{n \times n}$$

is said to be $*$-linear if $\mathcal{L}(X^*) = \mathcal{L}(X)^*$ for each $X \in \mathbb{F}^{q \times q}$.

Matrix representation 1: The Choi matrix $\mathbb{L}$ associated with $\mathcal{L}$ is given by

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$$(z \otimes x)^* \mathbb{L} (z \otimes x) \geq 0, \quad x \in \mathbb{F}^n, \; z \in \mathbb{F}^q.$$

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$$L : \mathbb{F}^{q^2} \to \mathbb{F}^{n^2}, \quad L\left(\operatorname{vec}(V)\right) = \operatorname{vec}(\mathcal{L}(V)), \quad V \in \mathbb{F}^{q\times q},$$

with vec the vectorization operator.

**Theorem.** [Hill 1973, Poluikis-Hill 1981] *A linear matrix map $\mathcal{L} : \mathbb{F}^{q\times q} \to \mathbb{F}^{n\times n}$ is $*$-linear (resp. completely positive) if and only if there exists $\mathbb{H}_{kl} \in \mathbb{F}$ and $C_1, \ldots, C_m \in \mathbb{F}^{n\times q}$, that are linearly independent, such that $\mathcal{L}$ is of the form*

$$\mathcal{L}(V) = \sum_{k,l=1}^{m} \mathbb{H}_{kl}\, C_l V C_k^*, \quad V \in \mathbb{F}^{q\times q} \tag{0.1}$$

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- We call (0.1) a minimal Hill representation of $\mathcal{L}$ and $\mathbb{H}$ its Hill matrix.
- The matricization $L$ and Choi matrix $\mathbb{L}$ of $\mathcal{L}$ take the form

$$L = \sum_{k,l=1}^{m} \mathbb{H}_{kl}\, \overline{C}_k \otimes C_l \quad \text{and} \quad \mathbb{L} = \widehat{C}^* \mathbb{H}^T \widehat{C},$$

  where $\widehat{C}^* := [\mathrm{vec}\,(C_1)\ \ldots\ \mathrm{vec}\,(C_m)] \in \mathbb{F}^{nq\times m}$ has full column rank. Furthermore, decomposing $L = [L_{ij}]$ with $L_{ij} \in \mathbb{F}^{n\times q}$ we have

$$\mathrm{span}\{C_1, \ldots, C_m\} = \mathrm{span}\{L_{ij} \colon i = 1, \ldots, n,\ j = 1, \ldots, q\}.$$

- We have $m = \mathrm{rank}\,\mathbb{L}$ and $\mathbb{H}$ is invertible.
- Minimal Hill representations are unique up to a change of basis of $\mathbb{F}^m$.

**Proposition.** [Klep et al. 2019] *The $*$-linear map $\mathcal{L} : \mathbb{F}^{q\times q} \to \mathbb{F}^{n\times n}$ is positive if and only if the Choi matrix $\mathbb{L}$ of $\mathcal{L}$ satisfies*

$$(z \otimes x)^* \, \mathbb{L} \, (z \otimes x) \geq 0, \qquad x \in \mathbb{F}^n, \, z \in \mathbb{F}^q.$$

Note that

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Thus positivity of $\mathcal{L}$ is equivalent to $y^* \mathbb{H}^T y \geq 0$ for all $y$ from the set

$$\mathfrak{Y}_{\widehat{C}} := \left\{ y_{z,x} = \widehat{C}(z \otimes x) : x \in \mathbb{F}^n, \, z \in \mathbb{F}^q \right\}.$$

Whether a positive $\mathcal{L}$ is also completely positive thus depends on the set $\mathfrak{Y}_{\widehat{C}}$ containing enough vectors to conclude $\mathbb{H} \in \overline{\mathcal{P}}_m$ from $y^* \mathbb{H}^T y \geq 0$ for all $y \in \mathfrak{Y}_{\widehat{C}}$.

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### Some sufficient conditions

(1) Yes, if $\mathfrak{Y}_{\widehat{C}} = \mathbb{F}^m$;

(2) Yes, if there exists a $x\in\mathbb{F}^n$ so that $\widehat{C}(I_q\otimes x)$ has full row-rank;

(3) Yes, if there exists a $z\in\mathbb{F}^q$ so that $\widehat{C}(z\otimes I_n)$ has full row-rank.

Note that the 1$^{\text{st}}$ is implied by the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ condition since

$$\widehat{C}(z\otimes x) = \widehat{C}(I_q\otimes x)z = \widehat{C}(z\otimes I_n)x.$$

## General analysis of 3rd condition



**Theorem.** *Let* $\mathcal{L} : \mathbb{F}^{q\times q} \to \mathbb{F}^{n\times n}$ *be* $*$*-linear, with matricization* $L = [L_{ij}]$ *and Choi matrix* $\mathbb{L}$*. Set*

$$\mathcal{W} := \mathrm{span}\{L_{ij} \colon\ i = 1, \ldots, n,\ j = 1, \ldots, q\}.$$

*Then for any minimal Hill representation of* $\mathcal{L}$*, for* $\widehat{C}$ *defined as before there exists a vector* $z \in \mathbb{F}^q$ *such that* $\widehat{C}(z \otimes I_n)$ *has full row-rank if and only if*

$$\text{(C1)}\quad \textit{For any linearly independent } X_1, \ldots, X_k \textit{ in } \mathcal{W},\ \textit{there exists a } v \in \mathbb{F}^q \\ \textit{such that } X_1 v, \ldots, X_k v \textit{ is linearly independent in } \mathbb{F}^n.$$

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**Lemma.** *The subspaces of upper triangular and lower triangular Toeplitz matrices in* $\mathbb{F}^{n\times n}$ *satisfy condition* (C1)*.*

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**Lemma.** *The subspaces of upper triangular and lower triangular Toeplitz matrices in $\mathbb{F}^{n\times n}$ satisfy condition* (C1).

**Lemma.** *Let $\mathcal{W}$ be a subspace of $\mathbb{F}^{n\times q}$ that satisfies the condition* (C1)*. Then:*

(i) *$\overline{\mathcal{W}}$ also satisfies* (C1);

(ii) *$P\mathcal{W}Q$ also satisfies* (C1) *for any invertible $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{q\times q}$;*

(iii) *any subspace $\mathcal{V}$ of $\mathcal{W}$ satisfies* (C1);

(iv) *if $\mathcal{Z}$ is a subspace of $\mathbb{F}^{n'\times q'}$ which satisfies* (C1)*, then $\mathcal{W} \oplus \mathcal{Z}$ is a subspace in $\mathbb{F}^{(n+n')\times(q+q')}$ that satisfies* (C1).

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**Proposition.** *For any matrix* $T \in \mathbb{F}^{n\times n}$*,* $\{T\}''_{\mathbb{F}}$ *satisfies* (C1)*.*

# General analysis of 3rd condition

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**Theorem.** *Let $\mathcal{L} : \mathbb{F}^{q\times q} \to \mathbb{F}^{n\times n}$ be $*$-linear, with matricization $L$. If there exists a matrix $C \in \mathbb{F}^{n\times n}$ such that $L \in \overline{\{C\}''_{\mathbb{F}}} \otimes \{C\}''_{\mathbb{F}}$, then positivity of $\mathcal{L}$ and complete positivity of $\mathcal{L}$ coincide.*

The Lyapunov operator $\mathcal{L}_Y$ for $Y \in \mathbb{F}^{n \times n}$, is $*$-linear and has matricization $L_Y$ given by

$$L_Y = Y^T \otimes I_n + I_n \otimes Y^*.$$

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Therefore, for $A, B \in \mathbb{F}^{n\times n}$ with $A$ Lyapunov regular and $B \in \{A\}''_{\mathbb{F}}$, follows

$$L_A, L_B \in \overline{\{A^*\}''_{\mathbb{F}}} \otimes \{A^*\}''_{\mathbb{F}} \text{ and hence } L_{A,B} = L_B L_A^{-1} \in \overline{\{A^*\}''_{\mathbb{F}}} \otimes \{A^*\}''_{\mathbb{F}},$$

with $L_{A,B}$ the matricization of $\mathcal{L}_{A,B}$.

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### Conclusion

For $A, B \in \mathbb{F}^{n\times n}$ with $A$ Lyapunov regular and $B \in \{A\}''_{\mathbb{F}}$, whether $B$ Lyapunov dominates $A$ can be verified by checking positive definiteness of matrix $\mathbb{H}$ that can be constructed concretely from $A$ and $B$.

# Solution to the interpolation problem: suboptimal case



Let $A \in \mathbb{F}^{n\times n}$ be Lyapunov regular and $B \in \{A\}''_{\mathbb{F}}$, and determine a minimal Hill representation of $\mathcal{L}_{A,B} = \mathcal{L}_B \circ \mathcal{L}_A^{-1}$:

$$\mathcal{L}_{A,B}(V) = \sum_{k,l=1}^{m} \mathbb{H}_{kl} C_k V C_l^* = \mathbf{C}^*(\mathbb{H} \otimes V)\mathbf{C}, \quad V \in \mathbb{F}^{n\times n},$$

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with $\mathbf{C}^* = \begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$. Because of minimality, $\mathbb{H}$ is unique up to a transformation of the basis of $\mathbb{F}^m$, so in this case we will talk about *the* Hill-Pick matrix associated with $(A, B)$ and denote it as $\mathbb{H}_{A,B}$.

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with $\mathbf{C}^* = \begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$. Because of minimality, $\mathbb{H}$ is unique up to a transformation of the basis of $\mathbb{F}^m$, so in this case we will talk about *the* Hill-Pick matrix associated with $(A, B)$ and denote it as $\mathbb{H}_{A,B}$.
Recall that the NP problem for $(A, B)$ is called suboptimal if

$$\operatorname{rank} \mathbb{H}_{A,B} = \operatorname{rank} \mathbb{L}_{A,B} = \dim\{A\}''_{\mathbb{F}}.$$

**Theorem.** *Let $A \in \mathbb{R}^{n\times n}$ be Lyapunov regular and $B \in \{A\}''_{\mathbb{R}}$. Assume that* $\operatorname{rank} \mathbb{H}_{A,B} = \dim\{A\}''_{\mathbb{R}}$. *Then the following statements are equivalent:*

(i) *$f(A) = B$ for a function $f \in \mathcal{PRO}$;*

(ii) *$B \in \mathcal{C}(A)$;*

(iii) *$\mathcal{L}_{A,B}$ is a positive linear map (i.e., $A \leq_{\mathcal{L}} B$);*

(iv) *$\mathcal{L}_{A,B}$ is a completely positive linear map;*

(v) *$\mathbb{H}_{A,B}$ is a positive definite matrix.*

# Solution to the interpolation problem: suboptimal case



Let $A \in \mathbb{F}^{n\times n}$ be Lyapunov regular and $B \in \{A\}''_{\mathbb{F}}$, and determine a minimal Hill representation of $\mathcal{L}_{A,B} = \mathcal{L}_B \circ \mathcal{L}_A^{-1}$:

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We know: $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$. To do: $(v) \Rightarrow (i)$.

# A skew-Hermitian Douglas lemma



For $A, B \in \mathbb{F}^{n\times n}$ with $A$ Lyapunov regular and $X \in \mathbb{F}^{n\times n}$:

$$XB + B^*X = \mathcal{L}_B(X) = \mathcal{L}_B\mathcal{L}_A^{-1}(\mathcal{L}_A(X)) = \mathbf{C}^*(\mathbb{H}_{A,B}\otimes XA)\mathbf{C} + \mathbf{C}^*(\mathbb{H}_{A,B}\otimes A^*X)\mathbf{C}.$$

Now assume $\mathbb{H}_{A,B} \in \mathcal{P}_m$ and factor $\mathbb{H}_{A,B} = P^*P$ for $P \in \mathbb{F}^{m\times m}$ invertible. Set

$$L_R := \begin{bmatrix} R \\ (P\otimes R)\mathbf{C} \end{bmatrix} \text{ and } M_R := \begin{bmatrix} RB \\ -(P\otimes RA)\mathbf{C} \end{bmatrix}, \ R \in \mathbb{F}^{n\times n}.$$

In the above identity take $X = R'^*R$, then we obtain $M_{R'}^*L_R = -L_{R'}^*M_R$. For any finite collection $\mathbf{R} = \{R_1, \ldots, R_k\} \subset \mathbb{F}^{n\times n}$ set

$$L_{\mathbf{R}} = \begin{bmatrix} L_{R_1} & \cdots & L_{R_k} \end{bmatrix} \text{ and } M_{\mathbf{R}} = \begin{bmatrix} M_{R_1} & \cdots & M_{R_k} \end{bmatrix}.$$

Since $M_{R'}^*L_R = -L_{R'}^*M_R$ for all $R, R' \in \mathbb{F}^{n\times n}$, we also have

$$\boxed{M_{\mathbf{R}}^*L_{\mathbf{R}} = -L_{\mathbf{R}}^*M_{\mathbf{R}}} \quad \text{for any finite } \mathbf{R} \subset \mathbb{F}^{n\times n}.$$

**Proposition.** *Let $A, B \in \mathbb{F}^{n\times n}$ with $A$ Lyapunov regular and $B \in \mathcal{C}_{\mathcal{L}}(A)\bigcap\{A\}''_{\mathbb{F}}$. Then there exists a matrix $\widehat{S} \in \mathbb{F}^{(1+m)n\times(1+m)n}$ so that*

$$\mathrm{P}_1: \ \widehat{S}L_R = M_R \text{ for each } R \in \mathbb{F}^{n\times n} \iff \mathrm{P}_2: \ \operatorname{Ker} L_{\mathbf{R}} \subset \operatorname{Ker} M_{\mathbf{R}} \text{ for any finite } \mathbf{R} \subset \mathbb{F}^{n\times n}.$$

*Moreover, if $\widehat{S} \in \mathbb{F}^{(1+m)n\times(1+m)n}$ satisfying $P_1$ exists, then WLOG*

$$\widehat{S} = S \otimes I_n \quad \text{for} \quad -S^* = S \in \mathbb{F}^{(1+m)\times(1+m)}.$$

*Furthermore, it suffices to check $P_2$ for a single $\mathbf{R} \subset \mathbb{F}^{n\times n}$ with the property that*

$$\operatorname{Ran} L_{\mathbf{R}} = \operatorname{span}\left\{L_R x : \ R \in \mathbb{F}^{n\times n}, \ x \in \mathbb{F}^n\right\}.$$

**Lemma.** *Let $A, B \in \mathbb{F}^{n\times n}$ with $A$ Lyapunov regular and $B \in \mathcal{C}_{\mathcal{L}}(A) \bigcap \{A\}''_{\mathbb{F}}$. Assume that the matrices $C_1, \ldots, C_m \in \{A^*\}''_{\mathbb{F}}$ in the minimal Hill representation of $\mathcal{L}_{A,B}$ are such that*

$$\operatorname{span}\{C_1, \ldots, C_m, I_n\} = \{A^*\}''_{\mathbb{F}}.$$

*Then* $\operatorname{Ker} L_{\mathbf{R}} \subset \operatorname{Ker} M_{\mathbf{R}}$ *for all finite collections* $\mathbf{R} \subset \mathbb{F}^{n\times n}$*. In particular, this occurs in the suboptimal case.*

### Proof for $\mathbf{R} = \{R\}$

To show: $\operatorname{Ker} L_R \subset \operatorname{Ker} M_R$. Since $L_R = \begin{bmatrix} R \\ (P \otimes R)\mathbf{C} \end{bmatrix}$ and $M_R = \begin{bmatrix} RB \\ -(P \otimes RA)\mathbf{C} \end{bmatrix}$ and $P$ is invertible, $x \in \operatorname{Ker} L_R$ iff

$$Rx = 0, \quad RC_j^* x = 0, \quad j = 1, \ldots, m.$$

Then

$$RTx = 0 \quad \text{for all } T \in \operatorname{span}\{C_1^*, \ldots, C_m^*, I_n\} = \{A\}''_{\mathbb{F}}.$$

Since $\{A\}''_{\mathbb{F}}$ is an algebra and $B, C_1^*, \ldots, C_m^* \in \{A\}''_{\mathbb{F}}$ also

$$RBx = 0, \quad RAC_j^* x = 0, \quad j = 1, \ldots, m,$$

from which $L_R x = 0$ follows. $\square$

**Lemma.** *Let $A, B \in \mathbb{R}^{n\times n}$ with $A$ Lyapunov regular and $B \in \mathcal{C}_{\mathcal{L}}(A) \bigcap \{A\}''_{\mathbb{R}}$. Assume that there exists a matrix $S \in \mathbb{R}^{(1+m)\times(1+m)}$ so that*

$$(S \otimes I_n) L_R = M_R, \quad R \in \mathbb{R}^{n\times n}, \text{ and } S^T = -S.$$

*Decompose $S$ as*

$$S = \begin{bmatrix} 0 & \ell \\ -\ell^T & -M \end{bmatrix} \quad \text{for} \quad M = -M^T \in \mathbb{R}^{m\times m}, \quad \ell \in \mathbb{R}^{1\times m}$$

*and set*

$$f(z) = \ell(zI_m - M)^{-1}\ell^T.$$

*Then $f \in \mathcal{PRO}$ and $f(A) = B$.*

**Sketch of proof.** Restrict to $R = I_n$ and solve in $(S \otimes I_n) L_{I_n} = M_{I_n}$:

$$B = (\ell \otimes I_n)(P \otimes I_n)\,\mathbf{C}$$

$$(I_m \otimes A - M \otimes I_n)(P \otimes I_n)\,\mathbf{C} = \left(\ell^T \otimes I_n\right).$$

Further, $(I_m \otimes A - M \otimes I_n)$ is invertible and thus

$$B = (\ell \otimes I_n)(I_m \otimes A - M \otimes I_n)^{-1}\left(\ell^T \otimes I_n\right) = f(A).$$

That $f \in \mathcal{PRO}$ follows from basic realization theory. $\square$

This proves the remaining implication (v) $\Rightarrow$ (i).

Thank you.