Square roots of H-nonnegative matrices

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SAMS, 6 December 2022

Indefinite inner product

A function $[.\,,.]$ from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ is called an *indefinite inner product* in $\mathbb{C}^n$ if the following hold for all $x, y, z \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$:

(i) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ (linear in the first argument),

(ii) $[x, y] = \overline{[y, x]}$ (anti-symmetric),

(iii) and if $[x, y] = 0$ for all $y \in \mathbb{C}^n$, then $x = 0$ (nondegenerate).

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Furthermore, if B is an H-selfadjoint matrix, then $\sigma(B)$ is symmetric relative to the real axis.

Notation

Let the *Jordan normal form* of a square matrix B be given by $S^{-1}BS = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_s)$, where

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix},$$

for $\lambda_i \in \sigma(B)$.

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for $\lambda_i \in \sigma(B)$. Let Q_n denote the $n \times n$ *sip matrix*:

$$Q_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Canonical form of (A, H) where A is H-selfadjoint

Theorem 1

Let $A \in \mathbb{C}^{n\times n}$ be an H-selfadjoint matrix. Then there exists an invertible matrix S over $\mathbb{C}$ such that $S^{-1}AS$ and S^*HS have the form

$$\begin{aligned} S^{-1}AS = {} & J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \\ & \oplus [J_{k_{\alpha+1}}(\lambda_{\alpha+1}) \oplus J_{k_{\alpha+1}}(\overline{\lambda}_{\alpha+1})] \oplus \cdots \oplus [J_{k_\beta}(\lambda_\beta) \oplus J_{k_\beta}(\overline{\lambda}_\beta)] \end{aligned} \tag{1}$$

where $\lambda_1, \ldots, \lambda_\alpha$ are real and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ are nonreal with positive imaginary parts; and

$$S^*HS = \varepsilon_1 Q_{k_1} \oplus \cdots \oplus \varepsilon_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta} \tag{2}$$

where $\varepsilon_1, \ldots, \varepsilon_\alpha$ are ± 1. The canonical form given by (1) and (2) is uniquely determined by the pair (A, H), up to a permutation of diagonal blocks.

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Sign characteristic and Segre characteristic

Definition 2

Let B be an $n \times n$ complex H-selfadjoint matrix with a single real eigenvalue λ and geometric multiplicity of one. We say B has a *positive sign characteristic* (with respect to H) if (B, H) has canonical form $(J_n(\lambda), Q_n)$ and a *negative sign characteristic* (with respect to H) if (B, H) has canonical form $(J_n(\lambda), -Q_n)$.

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Definition 3

Let B be a square nilpotent matrix with Jordan form $J = \bigoplus_{i=1}^{r} J_{n_i}(0)$ and assume that $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_r > 0$, where n_i represents the length of a Jordan chain. The *Segre characteristic* of B is defined as

$$(n_1, n_2, n_3, \ldots, n_r).$$

Example 1

Let the following matrices be given

$$B = J_2(0) \oplus J_2(0) \oplus J_1(0) \oplus J_1(0) = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 0 & 1 & & \\ & & 0 & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix};$$

$$H = Q_2 \oplus Q_2 \oplus -Q_1 \oplus Q_1 = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & -1 & \\ & & & & & 1 \end{bmatrix}.$$

The matrix B is H-nonnegative and has Segre characteristic $(2, 2, 1, 1)$.

H-selfadjoint square roots of H-selfadjoint matrices

Theorem 4 ([van der Mee, Ran, Rodman, 1999])

Let B be an H-selfadjoint matrix. Then there exists an H-selfadjoint matrix A such that $A^2 = B$ if and only if the canonical form of (B, H) has the following properties:

(i) *the Jordan blocks corresponding to the negative eigenvalues exist in pairs and the two Jordan blocks in each pair have opposite sign characteristic;*

(ii) *the Jordan blocks corresponding to the zero eigenvalue can be written as $J^{(1)} \oplus J^{(2)} \oplus J^{(3)}$, where $J^{(1)}$ is a direct sum of pairs $J_{p_i}(0) \oplus J_{p_i}(0)$, $J^{(2)}$ is a direct sum of pairs $J_{p_i}(0) \oplus J_{p_i-1}(0)$ and $J^{(3)}$ is a direct sum of 1×1 blocks and where the blocks in each pair in $J^{(1)}$ have opposite sign characteristic and those in each pair in $J^{(2)}$ have the same sign characteristic.*

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The problem

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Let B be an H-nonnegative matrix.

- Finding necessary and sufficient conditions for the existence of a square root for B.
- Finding a description of all possible square roots of nilpotent H-nonnegative matrices.
- Extending the results to finding H-selfadjoint and H-nonnegative square roots of B.

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Characteristics of H-nonnegative matrices

Theorem 5 ([Gohberg, Lancaster, Rodman, 2005])

A matrix B is H-nonnegative if and only if the following hold:

(i) *B is H-selfadjoint;*

(ii) *B has a real spectrum;*

(iii) *the canonical form of the pair (B, H) is $(S^{-1}BS, S^*HS)$ which is given by*

$$S^{-1}BS = \bigoplus_{i=1}^{q} J_1(\lambda_i) \oplus \bigoplus_{i=1}^{r-q} J_1(\lambda_{q+i}) \oplus \bigoplus_{i=1}^{s} J_1(0) \oplus \bigoplus_{i=1}^{t} J_2(0),$$

$$S^*HS = \bigoplus_{i=1}^{q} Q_1 \oplus \bigoplus_{i=1}^{r-q} (-Q_1) \oplus \bigoplus_{i=1}^{s} \varepsilon_i Q_1 \oplus \bigoplus_{i=1}^{t} Q_2,$$

where $\lambda_1, \ldots, \lambda_q > 0$, $\lambda_{q+1}, \ldots, \lambda_r < 0$, and $\varepsilon_i = \pm 1$.

Conditions for existence of a square root

Theorem 6 (See e.g. [Cross and Lancaster, 1974])

For any nilpotent matrix B there exists a square root A if and only if there exists a reordering $n_1, n_2, \ldots, n_p$ for some p, of the Segre characteristic of B such that the difference $n_{2j} - n_{2j-1}$ of each pair is equal to zero or one for all $j = 1, \ldots, \frac{p}{2}$.

Conditions on H-nonnegative matrix

Write any H-nonnegative matrix B in Jordan normal form as $B = B_- \oplus B_0 \oplus B_+$, where B_- has a negative spectrum, B_+ a positive spectrum, and B_0 has only zero in the spectrum.

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Theorem 7

Let B_0 be a nilpotent H-nonnegative matrix. Then the following hold:

(i) *If B_0 has an even number of Jordan chains of length two, then B_0 has a square root.*

(ii) *If B_0 has an odd number of Jordan chains of length two, then B_0 has a square root if and only if B_0 has at least one Jordan chain of length one.*

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Example

Example 2

Let B be H-nonnegative, with $B = J_2(0) \oplus J_2(0) \oplus J_1(0) \oplus J_1(0)$ and $H = Q_2 \oplus Q_2 \oplus -Q_1 \oplus Q_1$. Then the Segre characteristic of B is $(2, 2, 1, 1)$. Now there exists three different reorderings or pairings which can each deliver a square root:

- $(2, 2), (1, 1)$
- $(2, 2), (1, 0), (1, 0)$
- $(2, 1), (2, 1)$

Descriptions of square roots

Theorem 8

Let B_0 be a nilpotent H-nonnegative matrix in canonical form. That is, $B_0 = \bigoplus_{i=1}^{t} J_2(0) \oplus \bigoplus_{i=1}^{s} J_1(0)$. Then the square roots of blocks with all possible Segre characteristic are given as:

1. *The square root of $J_1(0) = [0]$, which is associated with the pair $(1,0)$, is equal to $[0]$;*
2. *The square root of $J_1(0) \oplus J_1(0)$, which is associated with the pair $(1,1)$, is*

$$\begin{bmatrix} \alpha & \frac{-\alpha^2}{\beta} \\ \beta & -\alpha \end{bmatrix}, \text{ for any complex numbers } \alpha, \beta \text{ with } \beta \neq 0, \text{ or } \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \tag{3}$$

for any nonzero complex number α;

Descriptions of square roots

Theorem 8 continued

3 *The square root of* $J_2(0) \oplus J_1(0)$*, which is associated with the pair* $(2,1)$*, is*

$$\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \end{bmatrix}, \text{ for complex numbers } \alpha \text{ and } \beta \neq 0; \tag{4}$$

4 *The square root of* $J_2(0) \oplus J_2(0)$*, which is associated with* $(2,2)$*, is*

$$\begin{bmatrix} -\alpha_3 & -\alpha_4 & \frac{-\alpha_3^2}{\alpha_1} & \beta \\ 0 & -\alpha_3 & 0 & \frac{-\alpha_3^2}{\alpha_1} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & \alpha_1 & 0 & \alpha_3 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 0 & 0 & \gamma_2 \\ 0 & \frac{1}{\gamma_2} & 0 & -\gamma_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

where α_1, γ_2 *is nonzero, and* $\beta = \frac{1}{\alpha_1^2}(\alpha_1 + \alpha_3^2\alpha_2 - 2\alpha_1\alpha_3\alpha_4)$.

Example

Let B and H be given by

$$B = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad H = Q_2 \oplus -Q_1 = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right].$$

Then B is H-nonnegative. The matrix

$$A = \begin{bmatrix} 0 & 1 & i \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$$

is a square root of B, but is not H-selfadjoint (i.e. $HA \neq A^*H$).

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is a square root of B, but is not H-selfadjoint (i.e. $HA \neq A^*H$). The matrix A is however H'-selfadjoint where

$$H' = Q_2 \oplus Q_1 = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Existence of H-selfadjoint square roots

Recall that for an H-nonnegative matrix we denote by

- s the number of Jordan blocks at zero of size one,
- t the number of Jordan blocks at zero of size two.

For the following we denote by s_+ (s_-) the number of positive (resp. negative) entries in the vector $(\varepsilon_1, \ldots, \varepsilon_s)$, hence we have

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$$s = s_- + s_+.$$

Theorem 9 ([Janse van Rensburg, vS, Theron, Trunk])

An H-nonnegative matrix B in $\mathbb{C}^n$ has an H-selfadjoint square root if and only if B satisfies the following conditions.

1. $\sigma(B) \subset [0, \infty)$;
2. $s_+ \geq t$;

Existence of H-nonnegative square roots

Theorem 10 ([Janse van Rensburg, vS, Theron, Trunk])

An H-nonnegative matrix B in $\mathbb{C}^{n\times n}$ has an H-nonnegative square root if and only if the following properties hold:

(i) $\sigma(B) \subset [0, \infty)$;

(ii) B *has no Jordan blocks of size two at the eigenvalue zero, that is,* $t = 0$.

References

Y. Bolshakov, C.V.M. van der Mee, A.C.M. Ran, B. Reichstein, L. Rodman (1997)
Polar decompositions in finite dimensional indefinite scalar product spaces: General Theory
Linear algebra and its applications: 216, 91–141.

G.W. Cross and P. Lancaster (1974)
Square roots of complex matrices
Linear and Multilinear Algebra, **1** , 289–293.

I. Gohberg, P. Lancaster, L. Rodman (2005)
Indefinite Linear Algebra and Applications
Birkhäuser Verlag, Basel.

D.B. Janse van Rensburg, M. van Straaten, F. Theron, C. Trunk (2021)
Square roots of H-nonnegative matrices
Linear Algebra and its Applications, **621**, 29–49.

C.V.M. van der Mee, A.C.M. Ran, and L. Rodman (1999)
Stability of self-adjoint square roots and polar decompositions in indefinite scalar product spaces
Linear Algebra and its Applications, **302-303** , 77–104.

Thank you.

This work is based on research supported in part by the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS).