Equivalence after extension and Schur coupling for Fredholm operators

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Joint work with Niels Laustsen; and Miek Messerschmidt, Mark Roelands, Andre Ran, Marten Wortel

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Definition Let $U \in \mathscr{B}(\mathcal{X})$ and $V \in \mathscr{B}(\mathcal{Y})$ be bounded operators on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We say that:

- U and V are equivalent after extension, abbreviated EAE, if there exist Banach spaces $\mathcal{X}_0$ and $\mathcal{Y}_0$ and isomorphisms $E \in \mathscr{B}(\mathcal{Y} \oplus \mathcal{Y}_0, \mathcal{X} \oplus \mathcal{X}_0)$ and $F \in \mathscr{B}(\mathcal{X} \oplus \mathcal{X}_0, \mathcal{Y} \oplus \mathcal{Y}_0)$ such that
$$\begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{X}_0} \end{bmatrix} = E \begin{bmatrix} V & 0 \\ 0 & I_{\mathcal{Y}_0} \end{bmatrix} F.$$
- U and V are Schur coupled, abbreviated SC, if there exist invertible $A \in \mathscr{B}(\mathcal{X})$ and $D \in \mathscr{B}(\mathcal{Y})$ and operators $B \in \mathscr{B}(\mathcal{Y}, \mathcal{X})$ and $C \in \mathscr{B}(\mathcal{X}, \mathcal{Y})$ such that
$$U = A - BD^{-1}C \quad \text{and} \quad V = D - CA^{-1}B.$$

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Theorem [Bart-Tsekanovskii '92] *If U and V are SC, then they are also EAE.*

Question [BT '92] Is the converse also true, i.e., does EAE imply SC?

- Relevant for various applications, including integral equations.
- True for many special cases and classes of operators.
- tH-Messerschmidt-Ran-Roelands '19: No, when $\mathcal{X}$ and $\mathcal{Y}$ are *essentially incomparable*.

When do EAE and SC coincide?

Partially confirmative answers

- BT'94: Yes if U and V are Fredholm Hilbert space operators.
- BT'94: Yes if U and V are Fredholm operators with index 0.
- BGKR'05: Yes if SC is an equivalence relation (this is true for EAE).
- tH-Ran'13: Yes for operators on separable Hilbert spaces.
- Timotin'14: Yes for all Hilbert space operators. And a characterization for EAE in terms of spectral resolutions of $|U|$ and $|V|$.
- tH-Ran'13: Yes for operators in norm closure of the invertible operators.
- tHMRRW '18: Yes for compact operators, and some larger operator ideals.
- tHMR '20: $\mathcal{X}$ has complemented copy of $\mathcal{Y}$, or conversely, U, V Fredholm.

(BT=Bart-Tsekanovskii, BGKR=Bart-Gohberg-Kaashoek-Ran, tHMRRW=tH-Messerschmidt-Ran-Roelands-Wortel)

Definition An operator T in $\mathcal{B}(\mathcal{X},\mathcal{Y})$ is Fredholm if

$$\alpha(T) := \dim \operatorname{Ker} W < \infty \quad \text{and} \quad \beta(T) := \dim \mathcal{Y}/\operatorname{Im} T < \infty,$$

and in this case the index of T is defined as $\operatorname{Ind}(T) = \alpha(T) - \beta(T) \in \mathbb{Z}$.
We write $\Phi(\mathcal{X},\mathcal{Y})$ for the Fredholm operators in $\mathcal{B}(\mathcal{X},\mathcal{Y})$ and $\Phi_k(\mathcal{X},\mathcal{Y})$ for the Fredholm operators in $\mathcal{B}(\mathcal{X},\mathcal{Y})$ of index $k \in \mathbb{Z}$.
When $\mathcal{X} = \mathcal{Y}$: $\Phi(\mathcal{X}) := \Phi(\mathcal{X},\mathcal{X})$ and $\Phi_k(\mathcal{X}) := \Phi_k(\mathcal{X},\mathcal{X})$.

Definition Two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are *essentially incomparable* if $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every $S \in \mathscr{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathscr{B}(\mathcal{X}, \mathcal{Y})$.

Pitt-Rosenthal Theorem

Any operator $T \in \mathscr{B}(\ell^p, \ell^q)$, for $1 \leq q < p < \infty$, is compact.

Corollary *ℓ^p and ℓ^q, $p \neq q$, are essentially incomparable.*

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Theorem *Let $U \in \mathscr{B}(\mathcal{X})$ and $V \in \mathscr{B}(\mathcal{Y})$ with $\mathcal{X}$ and $\mathcal{Y}$ infinite dimensional, essentially incomparable Banach spaces. Then*

- *U and V are SC if and only if U and V are Fredholm with index 0 and $\dim\ker U = \dim\ker V$ (and thus $\dim\operatorname{coker} U = \dim\operatorname{coker} V$);*
- *U and V are EAE if and only if U and V are Fredholm with $\dim\ker U = \dim\ker V$ and $\dim\operatorname{coker} U = \dim\operatorname{coker} V$.*

Corollary *In general the operators relation SC and EAE do not coincide: Take for U and V both the forward shift on $\mathcal{X} = \ell^p$ and $\mathcal{Y} = \ell^q$, $p \neq q$.*

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Remark *For Fredholm operators on $\mathcal{X}$ and $\mathcal{Y}$, EAE and SC coincides when $\mathcal{X} \simeq \mathcal{Y}$, (more generally when one has a complemented copy in the other), while this conclusion can not be reached when $\mathcal{U}$ and $\mathcal{Y}$ are essentially incomparable.*

EAE for Fredholm operators

The equivalence relation EAE is well understood for Fredholm operators.

Theorem [Bart-Tsekanovskii '92] *Let $U \in \mathscr{B}(\mathcal{X})$ and $V \in \mathscr{B}(\mathcal{Y})$.*

- *If U and V are EAE, then: $U \in \Phi_k(\mathcal{X}) \Longleftrightarrow V \in \Phi_k(\mathcal{Y})$.*
- *If $U \in \Phi(\mathcal{X})$ and $V \in \Phi(\mathcal{Y})$, then:*

$$U \text{ and } V \text{ are EAE} \quad \Longleftrightarrow \quad \alpha(U) = \alpha(V) \text{ and } \beta(U) = \beta(V).$$

In particular, $\operatorname{Ind}(U) = \operatorname{Ind}(V)$ in case U and V are EAE Fredholm operators.

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In particular, $\mathrm{Ind}(U) = \mathrm{Ind}(V)$ *in case U and V are EAE Fredholm operators.*

Proposition *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ define*

$$\begin{aligned}\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) &= \{(U,V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : U \text{ and } V \text{ are EAE}\} \\ &= \{(U,V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : \alpha(U) = \alpha(V)\}.\end{aligned}$$

Then

$$\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset \quad \Leftrightarrow \quad \Phi_k(\mathcal{X}) \neq \emptyset \quad \text{and} \quad \Phi_k(\mathcal{Y}) \neq \emptyset.$$

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- *If* U *and* V *are EAE, then:* $U \in \Phi_k(\mathcal{X}) \iff V \in \Phi_k(\mathcal{Y})$.
- *If* $U \in \Phi(\mathcal{X})$ *and* $V \in \Phi(\mathcal{Y})$*, then:*

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Remark *In '94 Gowers constructed a Banach space* $\mathcal{X}$ *that was not isomorphic to any of its hyperplanes, which implies that*

$$\mathbb{I}_\Phi(\mathcal{X}) := \{k \in \mathbb{Z} \colon \Phi_k(\mathcal{X}) \neq \emptyset\} = \{0\}.$$

In '97 Gowers and Maurey, for each $k_0 \in \mathbb{Z}$ *constructed a Banach space* $\mathcal{X}$ *so that* $\mathbb{I}_\Phi(\mathcal{X}) = k_0\mathbb{Z}$.

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- *If $U \in \Phi(\mathcal{X})$ and $V \in \Phi(\mathcal{Y})$, then:*

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Then

$$\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset \iff \Phi_k(\mathcal{X}) \neq \emptyset \quad \text{and} \quad \Phi_k(\mathcal{Y}) \neq \emptyset.$$

Proposition *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ define*

$$\mathbb{I}_{\mathrm{EAE}}(\mathcal{X},\mathcal{Y}) = \{k \in \mathbb{Z} : \mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset\}.$$

Then $\mathbb{I}_{\mathrm{EAE}}(\mathcal{X},\mathcal{Y}) = \mathrm{eae}(\mathcal{X},\mathcal{Y})\,\mathbb{Z}$ *where*

$$\mathbb{N}_0 \ni \mathrm{eae}(\mathcal{X},\mathcal{Y}) := \begin{cases} 0 & \text{if } \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) \cap \mathbb{N} = \emptyset,\\ \min \mathbb{I}_\Phi(\mathcal{X}) \cap \mathbb{I}_\Phi(\mathcal{Y}) \cap \mathbb{N} & \text{otherwise.}\end{cases}$$

For which $k \in \mathbb{Z}$ do we have $\mathrm{EAE}_k(\mathcal{X}, \mathcal{Y}) = \mathrm{SC}_k(\mathcal{X}, \mathcal{Y})$?

Definition For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, set

$$\begin{aligned} \mathrm{EAE}_k(\mathcal{X}, \mathcal{Y}) &= \{(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : U \text{ and } V \text{ are EAE}\}, \\ \mathrm{SC}_k(\mathcal{X}, \mathcal{Y}) &= \{(U, V) \in \Phi_k(\mathcal{X}) \times \Phi_k(\mathcal{Y}) : U \text{ and } V \text{ are SC}\}. \end{aligned}$$

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Theorem *The following four conditions are equivalent for every pair of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and every $k \in \mathbb{Z}$:*

- *There exists a pair of operators $S \in \mathscr{B}(\mathcal{Y},\mathcal{X})$ and $T \in \mathscr{B}(\mathcal{X},\mathcal{Y})$ such that $I_{\mathcal{X}} - ST \in \Phi_k(\mathcal{X})$.*
- *There exists a pair of operators $S \in \mathscr{B}(\mathcal{Y},\mathcal{X})$ and $T \in \mathscr{B}(\mathcal{X},\mathcal{Y})$ such that $I_{\mathcal{Y}} - TS \in \Phi_k(\mathcal{Y})$.*
- $\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) = \mathrm{SC}_k(\mathcal{X},\mathcal{Y})$*, and this set is non-empty.*
- $\mathrm{SC}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset$*.*

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- $\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) = \mathrm{SC}_k(\mathcal{X},\mathcal{Y})$, *and this set is non-empty.*
- $\mathrm{SC}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset$.

Corollary *For every pair of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and every $k \in \mathbb{Z}$:*

$$\mathrm{SC}_k(\mathcal{X},\mathcal{Y}) = \emptyset \quad \text{or} \quad \mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) = \mathrm{SC}_k(\mathcal{X},\mathcal{Y}).$$

If $k \in \mathrm{eae}(\mathcal{X},\mathcal{Y})\mathbb{Z}$, so that $\mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset$, precisely one of the two occurs.

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Corollary *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ set*

$$\mathbb{I}_{\mathrm{EAE}}(\mathcal{X},\mathcal{Y}) = \{k \in \mathbb{Z} : \mathrm{EAE}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset\}, \ \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) = \{k \in \mathbb{Z} : \mathrm{SC}_k(\mathcal{X},\mathcal{Y}) \neq \emptyset\}.$$

Then EAE and SC coincide for Banach spaces iff $\mathbb{I}_{\mathrm{EAE}}(\mathcal{X},\mathcal{Y}) = \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$.

Do we have $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) = \mathrm{sc}(\mathcal{X},\mathcal{Y})\mathbb{Z}$ for some $\mathrm{sc}(\mathcal{X},\mathcal{Y}) \in \mathbb{N}_0$?

The obvious candidate for $\mathrm{sc}(\mathcal{X},\mathcal{Y})$ is:

$$\mathrm{sc}(\mathcal{X},\mathcal{Y}) = \begin{cases} 0 & \text{if } \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) \cap \mathbb{N} = \emptyset, \\ \min \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) \cap \mathbb{N} & \text{otherwise.} \end{cases}$$

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Lemma *The set $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$ has the following properties:*

- *0 and $\mathrm{sc}(\mathcal{X},\mathcal{Y})$ are in $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$.*
- *$k\mathbb{Z} \subset \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$ for every $k \in \mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$.*

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Corollary $\mathbb{I}_{SC}(\mathcal{X},\mathcal{Y}) = \mathrm{sc}(\mathcal{X},\mathcal{Y})\mathbb{Z}$ *iff* $\mathbb{I}_{SC}(\mathcal{X},\mathcal{Y})$ *is closed under addition.*

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- An operator $T \in \mathscr{B}(\mathcal{X},\mathcal{Y})$ is inessential if $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every $S \in \mathscr{B}(\mathcal{Y},\mathcal{X})$; we write $\mathscr{E}(\mathcal{X},\mathcal{Y})$ for the inessential operators in $\mathscr{B}(\mathcal{X},\mathcal{Y})$.
- We say that a subset Σ of $\mathscr{B}(\mathcal{X},\mathcal{Y})$ is *essentially closed under addition* if, for every pair of operators $U, V \in \Sigma$, there exists an inessential operator $R \in \mathscr{E}(\mathcal{X},\mathcal{Y})$ such that $U + V - R \in \Sigma$.
- Set $\mathscr{G}_{\mathcal{Y}}(\mathcal{X}) = \{ST : S \in \mathscr{B}(\mathcal{Y},\mathcal{X}),\ T \in \mathscr{B}(\mathcal{X},\mathcal{Y})\}$.

Do we have $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) = \mathrm{sc}(\mathcal{X},\mathcal{Y})\mathbb{Z}$ for some $\mathrm{sc}(\mathcal{X},\mathcal{Y}) \in \mathbb{N}_0$?

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- We say that a subset Σ of $\mathscr{B}(\mathcal{X},\mathcal{Y})$ is essentially closed under addition if, for every pair of operators $U, V \in \Sigma$, there exists an inessential operator $R \in \mathscr{E}(\mathcal{X},\mathcal{Y})$ such that $U + V - R \in \Sigma$.
- Set $\mathscr{G}_{\mathcal{Y}}(\mathcal{X}) = \{ST : S \in \mathscr{B}(\mathcal{Y},\mathcal{X}),\ T \in \mathscr{B}(\mathcal{X},\mathcal{Y})\}$.

Proposition *Suppose that at least one of the sets $\mathscr{G}_{\mathcal{Y}}(\mathcal{X})$ and $\mathscr{G}_{\mathcal{X}}(\mathcal{Y})$ is essentially closed under addition. Then $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$ is closed under addition.*

Do we have $\mathbb{I}_{\mathsf{SC}}(\mathcal{X},\mathcal{Y}) = \mathrm{sc}(\mathcal{X},\mathcal{Y})\mathbb{Z}$ for some $\mathrm{sc}(\mathcal{X},\mathcal{Y}) \in \mathbb{N}_0$?

The obvious candidate for $\mathrm{sc}(\mathcal{X},\mathcal{Y})$ is:

$$\mathrm{sc}(\mathcal{X},\mathcal{Y}) = \begin{cases} 0 & \text{if } \mathbb{I}_{\mathsf{SC}}(\mathcal{X},\mathcal{Y}) \cap \mathbb{N} = \emptyset, \\ \min \mathbb{I}_{\mathsf{SC}}(\mathcal{X},\mathcal{Y}) \cap \mathbb{N} & \text{otherwise.} \end{cases}$$

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- For the quasi-reflexive James spaces $\mathcal{J}_p$ and cJ_q, $p \neq q$, $\mathcal{X} = \mathcal{J}_p \oplus \mathcal{J}_p \oplus \mathcal{J}_q$ and $\mathcal{Y} = \mathcal{J}_p \oplus \mathcal{J}_q \oplus \mathcal{J}_q$ are such that $\mathscr{G}_{\mathcal{Y}}(\mathcal{X})$ and $\mathscr{G}_{\mathcal{X}}(\mathcal{Y})$ are not essentially closed under addition, yet $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y})$ is closed under addition.

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Proposition *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ we have:*

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Corollary *For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ we have $\mathbb{I}_{\mathrm{SC}}(\mathcal{X},\mathcal{Y}) = \mathbb{I}_{\mathrm{EAE}}(\mathcal{X},\mathcal{Y})$ if and only if $\mathrm{sc}(\mathcal{X},\mathcal{Y}) = \mathrm{eae}(\mathcal{X},\mathcal{Y})$.*

Essential and projectively incomparability

Definition Two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are called

- *essentially incomparable* if $I_{\mathcal{X}} - ST \in \Phi(\mathcal{X})$ for every $S \in \mathscr{B}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathscr{B}(\mathcal{X}, \mathcal{Y})$;
- *totally incomparable* if no closed, infinite-dimensional subspace of $\mathcal{X}$ embeds isomorphically into $\mathcal{Y}$;
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Theorem

- *For every* $k_0 \in \mathbb{N}$*, there exists a pair of projectively incomparable Banach spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *such that* $\mathrm{sc}(\mathcal{X}, \mathcal{Y}) = \mathrm{eae}(\mathcal{X}, \mathcal{Y}) = k_0$*.*
- *For every* $k_0 \in \mathbb{N}$*, there exists a pair of projectively incomparable Banach spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *such that* $\mathrm{eae}(\mathcal{X}, \mathcal{Y}) = 1$ *and* $\mathbb{I}_{\mathrm{SC}}(\mathcal{X}, \mathcal{Y}) = k_0\mathbb{Z}$*, and consequently* $\mathrm{sc}(\mathcal{X}, \mathcal{Y}) = k_0$*.*

When $\mathcal{X}$ and $\mathcal{Y}$ are not projectively incomparability

When $\mathcal{X}$ and $\mathcal{Y}$ are not projectively incomparability, then

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- *Answer: No. Take $\mathcal{X} = \left(\bigoplus_{n=1}^{\infty} \ell_1^n\right)_{c_0}$ and $\mathcal{Y} = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0}$. Both contain c_0 as complemented copy. For $\mathcal{Z} = \mathcal{X}$ or $\mathcal{Z} = \mathcal{Y}$, if $\mathcal{W}$ is a complemented subspace of $\mathcal{Z}$, then $\mathcal{W} \simeq c_0$ or $\mathcal{W} \simeq \mathcal{Z}$.*

THANK YOU FOR YOUR ATTENTION