Extreme point methods in the study of isometries on certain noncommutative spaces

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- The aim of this talk is to present a non-commutative analogue of this result.

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Outline

1. $L^1 + L^\infty$-spaces
   - Commutative $L^1 + L^\infty$-spaces
   - Non-commutative $L^1 + L^\infty$-spaces
2. Extreme points of the unit balls of $L^1 + L^\infty$-spaces
   - Extreme points of the unit balls of commutative $L^1 + L^\infty$-spaces
   - Extreme points of the unit balls of non-commutative $L^1 + L^\infty$-spaces
3. Isometries on $L^1 + L^\infty$-spaces
   - Isometries on commutative $L^1 + L^\infty$-spaces
   - Isometries on non-commutative $L^1 + L^\infty$-spaces

Commutative L^1+L^∞ spaces

Ingredients:

- (Ω,Σ,μ) - a measure space
- $L(\Omega)$ - the space of all (equivalence classes of) measurable functions on Ω
- $L^0_\infty(\Omega)$ - functions in $L(\Omega)$ which are bounded, except possibly on a set of finite measure
- The linear space $L^1+L^\infty(\Omega)=\{f\in L^0_\infty(\Omega): f=g+h, g\in L^1(\Omega), h\in L^\infty(\Omega)\}$
- Equipped with the norm

$$\left\|f\right\|_{L^1+L^\infty(\Omega)}=\inf\{\left\|g\right\|_1+\left\|h\right\|_\infty : f=g+h, g\in L^1(\Omega), h\in L^\infty(\Omega)\}$$

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The non-commutative setting

Ingredients:

- von Neumann algebra: $\mathscr{A} \subseteq B(H)$ (we will stick to semi-finite ones!)
- Faithful normal semi-finite (fns) trace: $\tau : \mathscr{A}^+ \to [0, \infty]$
- τ-measurable operators: $L^0_\infty(\mathscr{A})$ (also denoted $S(A, \tau)$ or $S(\tau)$).
  - closed, densely defined operators
  - affiliated with the von Neumann algebra
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Example

- Let (Ω, Σ, μ) be a σ-finite measure space and $H = L^2(\Omega)$.
- For $f \in L(\Omega)$, define

$$M_f(g) = f.g \qquad \forall g \in L^2(\mu) : fg \in L^2(\mu)$$

- Let $\mathscr{A} = \{M_f : f \in L^\infty(\mu)\}$
- Defining a trace:

$$\tau(M_f) := \int_\Omega f \, d\mu \qquad \forall M_f \in \mathscr{A}^+$$

- The set of trace-measurable operators:

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Proposition (Hudzik (1993))

Suppose (Ω, Σ, μ) is a nonatomic measure space. If $\mu(\Omega) \leq 1$, then the unit ball of $L^1 + L^\infty(\Omega)$ does not have any extreme points. If $\mu(\Omega) > 1$, then f is an extreme point of the unit ball of $L^1 + L^\infty(\Omega)$ if and only if $|f(t)| = 1$ for μ-a.e. $t \in \Omega$.

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Isometries on commutative $L^1 + L^\infty$-spaces

Theorem (Grząślewicz (1992(b)))

Let T be a surjective isometry of $L^1 + L^\infty[0,\infty)$. Then T is of the form

$$(Tf)(t) = r(t)f(\phi(t)),$$

where $|r(t)| = 1$ m-a.e and $\phi : [0,\infty) \to [0,\infty)$ is an invertible measure preserving transformation.

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Suppose $(\mathscr{A},\tau)$ and $(\mathscr{B},\nu)$ are non-atomic semi-finite von Neumann algebras with $\tau(1),\nu(1) > 1$. If $U : L^1 + L^\infty(\mathscr{A}) \to L^1 + L^\infty(\mathscr{B})$ is a surjective isometry, then there exist a unique trace-preserving Jordan $*$-isomorphism J from $S(\mathscr{A},\tau)$ onto $S(\mathscr{B},\nu)$ and a unitary operator $u = U(1) \in \mathscr{B}$ such that $U(x) = uJ(x)$ for all $x \in L^1 + L^\infty(\mathscr{A})$. Furthermore, $U\restriction_{\mathscr{A}}$ is an L^∞-isometry from $\mathscr{A}$ onto $\mathscr{B}$ and $U\restriction_{L^1(\mathscr{A})}$ is an L^1-isometry from $L^1(\mathscr{A})$ onto $L^1(\mathscr{B})$. Conversely, if $J : S(\mathscr{A},\tau) \to S(\mathscr{B},\nu)$ is a trace-preserving Jordan $*$-isomorphism and $u \in \mathscr{B}$ is a unitary operator, then letting $U(x) = uJ(x)$ for $x \in L^1 + L^\infty(\mathscr{A})$ yields a surjective isometry from $L^1 + L^\infty(\mathscr{A})$ onto $L^1 + L^\infty(\mathscr{B})$.

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Suppose U is a surjective isometry from $L^1 + L^\infty(\mathscr{A})$ onto $L^1 + L^\infty(\mathscr{B})$.

- Suppose $x \in B_{\mathscr{A}}$. Then $x = \frac{1}{2}(v + w)$ for some $v, w \in \operatorname{ext}(B_{\mathscr{A}})$, by [Theorem 3, Choda (1970)].
- It follows from the characterization of the extreme points that the unit ball of $\mathscr{A}$ and the unit ball of $L^1 + L^\infty(A)$ have the same extreme points
    - Therefore $U(v)$ and $U(w)$ are extreme points of $B_{\mathscr{B}}$.
    - One can use this to show that U maps $\mathscr{A}$ onto $\mathscr{B}$ L^∞-isometrically
    - By Kadison's Theorem [Kadison (1951)], there exists a Jordan $*$-isomorphism from $\mathscr{A}$ onto B and a unitary operator $u \in B$ such that $U(x) = uJ(x)$ for all $x \in \mathscr{A}$.
- Extend this representation of U to all of $L^1 + L^\infty(\mathscr{A})$.

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- If $\tau(1_{\mathscr{A}}) < \infty$, then $L^1 + L^\infty(\mathscr{A}) = L^1(\mathscr{A})$.
- If, in addition $\tau(1_{\mathscr{A}}) \leq 1$, then $\|x\|_{1+\infty} = \|x\|_1$ for every $x \in L^1 + L^\infty(\mathscr{A})$.
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The End

Thank you for your attention.

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