On convex structures in quasi-metric spaces

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SOUTH AFRICAN MATHEMATICAL SOCIETY
(SAMS 2022)
06 DECEMBER 2022 - 08 DECEMBER 2022
Stellenbosch University

Introduction

In 1970, Takahashi introduced the notion of convexity in metric spaces. A convex metric space is a generalized space. Recently Kunzi and Yildiz initiated the study on convex structures in the sense of Takahashi in T_0-quasi-metric spaces. They considered a T_0-quasi-metric space (X, q) equipped with a Takahashi convexity structure. In this talk, we

generalize and extend the classical fixed point theorems. Moreover, we study the concept of fixed point theorems for nonexpansive mappings on convex T_0- quasi-metric spaces.

Preliminaries

Definition

Let X be a set and let $q : X \times X \to [0, \infty)$ be a function. Then q is called a quasi-metric on X if

(i) $q(x, x) = 0$ *for all* $x \in X$

(ii) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$

Furthermore, q is a T_0-quasi-metric if

$$q(x, y) = 0 = q(y, x) \text{ implies that } x = y,$$

for each $x, y \in X$. Then the pair (X, q) is called a quasi-metric space.

We shall say that q is a T_0-quasi-metric provided that q satisfies the following condition: for each $x, y \in X$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$.

Remark

If q is a quasi-metric on a set X, then $q^{-1} : X \times X \to [0, \infty)$ on X defined by $q^{-1}(x, y) = q(y, x)$ for every $x, y \in X$, is called the conjugate quasi-metric. A quasi-metric on a set X such that $q = q^{-1}$ is a metric. Note that if (X, q) is a T_0-quasi-metric space, then $q^s = \sup\{q, q^{-1}\} = q \vee q^{-1}$ is a metric.

Example

For $a, b \in \mathbb{R}$ we shall put $a \dot{-} b = \max\{a - b, 0\}$. If we equip $\mathbb{R}$ with $u(a, b) = a \dot{-} b$, then $(\mathbb{R}, u)$ is a T_0-quasi-metric space that we call the standard T_0-quasi-metric of $\mathbb{R}$. Note that the symmetrize metric u^s of u is the usual metric on $\mathbb{R}$ where $u^s(a, b) = |a - b|$ whenever $a, b \in \mathbb{R}$.

Definition

Let X be a real vector space. A function $\| \cdot |: X \to [0, \infty)$ is called an asymmetric seminorm *on X if for any $x, y \in X$ and $t \in [0, \infty)$ we have:*

(a) $\| tx | = t \| x |$;
(b) $\| x + y | \leq \| x | + \| y |$.

If in addition
(c) $\| x | = \| -x | = 0$ *if and only if* $x = 0$,

holds then $\| \cdot |$ is called an asymmetric norm, and the pair $(X, \| \cdot |)$ is called an asymmetrically normed space.

Each asymmetrically normed vector space induces a T_0- quasi-metric space $q_{\| \cdot |}$ on X by setting $q_{\| \cdot |}(x, y) = \| x - y |$ whenever $x, y \in X$.

Example

Let $\mathbb{R}$ *be equipped with its usual real vector space structure. Then* $\| x |_u = x$ *if* $x \geq 0$ *and* $\| x |_u = 0$ *otherwise, defines an asymmetric norm on* $\mathbb{R}$ *with the induced* T_0- *quasi-metric* u, *as defined above.*

Example

Let (X, q) *be a* T_0-*quasi-metric space,* BX *be the real vector space of bounded real-valued functions on* X *and*

$$\| f |= \sup_{x \in X}(f(x) - 0)$$

whenever $f \in X$. *Clearly* $\| f |$ *is an asymmetric norm on* BX. *Consequently* D *defined by the condition that*

$$D(f, g) = \| f - g |$$

whenever $f, g \in BX$ *is a* T_0-*quasi-metric on* BX.

Convex structures in quasi-metric spaces

Definition

Let (X, q) *be a quasi-metric space. A mapping* $W : X \times X \times \to [0, 1]$ *is said to be a* convex structure *on* X *if for all* $x, y \in X$ *and* $\lambda \in [0, 1]$,

$$q(x, W(x, y, \lambda)) \leq \lambda q(z, x) + (1 - \lambda)q(x, y),$$

and

$$q(W(x, z, \lambda), x) \leq \lambda q(x, z) + (1 - \lambda)q(y, z)$$

whenever $z \in X$.

Then (X, q) *equipped with a convex structure is said to be a* convex quasi-metric space *denoted by* (X, q, W).

Example

Let $\mathbb{R}$ be the set of real numbers be equipped with the standard T_0-quasi-metric space

$$q(x, y) = \max\{0, x - y\},$$

whenever $x, y \in \mathbb{R}$. If we define

$$W(x, y, \lambda) = \lambda x + (1 - \lambda)y$$

whenever $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, then $(\mathbb{R}, q, W)$ is a convex quasi-metric space.

Proposition

Suppose that (X, q, W) *is a convex* T_0*-quasi-metric space, then* (X, q^s, W) *is a convex metric space.*

Remark

For any convex T_0*- quasi-metric space* (X, q, W), *the following are true:*

(1) For any $x \in X$ *and* $\lambda \in [0, 1]$, *we have* $W(x, x, \lambda) = x$.

(2) For any $x, y \in X$, *it follows that* $W(y, x, 0) = x$ *and* $W(y, x, 1) = y$.

Definition (Kunzi and Yildiz)

Let X be a vector space.
(a) The convex structure W is called translation-invariant if W satisfies the condition

$$W(x+z, y+z, \lambda) = W(x, y, \lambda) + z$$

for all $x, y, z \in X$ and $\lambda \in [0, 1]$.

(b) We say that the convex structure satisfies the homogeneity condition if for any $\alpha \in \mathbb{R}$ we have

$$W(\alpha x, \alpha y, \lambda) = \alpha W(x, y, \lambda)$$

for any $x, y \in X$ and $\lambda \in [0, 1]$.

Fixed points in convex T_0-quasi-metric spaces

Definition

A convex T_0-quasi-metric space (X, q, W) is said to have property (H) if any decreasing family $\{D_i\}_{i \in I}$ of nonempty doubly closed convex bounded subsets of X such that $D_j \subset D_i$ with $i \leq j$, has nonempty intersection.

Theorem

Let W be the unique convex structure on a T_0-quasi metric space (X, q) with the property (H). If K is a nonempty doubly closed convex bounded subset of X with the normal structure, then any commuting family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K, q) has a nonempty common fixed point set $(i.e. \cap_{i=1}^{n} Fix(T_i) \neq \emptyset)$,where $Fix(T_i)$ denotes the set of fixed points of T_i, that is
$Fix(T_i) = \{x \in K : T_i(x) = x\}$.

Theorem

Let (X, q, W) be a convex T_0-quasi metric space and K be a nonempty doubly closed convex bounded subset of X with the normal structure. If $T : (K, q) \to (K, q)$ is a nonexpansive map, then T has fixed point.

W-convex function pairs and Isbell-hull

Let $(X, \| \cdot |)$ be an asymmetrically normed real vector space and let a pair of functions $f = (f_1, f_2)$, where $f_j : X \to \mathbb{R}$ for $j = 1, 2$. The pair of functions $f = (f_1, f_2)$ is called ample on X if $\| x - y | \leq f_2(x) + f_1(y)$ for all $x, y \in X$.

Moreover, the pair of functions $f = (f_1, f_2)$ is called minimal if for any ample pair of functions $g = (g_1, g_2)$ on X such that $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ for all $x \in X$, then $g_1 = f_1$ and $g_2 = f_2$.

The set of all minimal pairs of functions on X is denoted by $\varepsilon(X, \| \cdot |)$ and it is called the Isbell-hull of $(X, \| \cdot |)$. Note that the Isbell-hull of an asymmetrically normed real vector space is 1-injective and Isbell-convex. If $f = (f_1, f_2) \in \varepsilon(X, \| \cdot |)$, then it is well-known that for any $x \in X$,

$$f_1(x) = \sup_{z \in X} u[\| x - z | - f_2(x)]$$

Definition

Let $(X, \|\cdot|)$ *be an asymmetrically normed real vector space. We say that* $(X, \|\cdot|, W)$ *is* ***convex asymmetrically normed real vector space****, if* W *is convex structure on the quasi-metric space* $(X, q_{\|\cdot|})$.

Definition

Let $(X, \|\cdot|, W)$ *be a convex asymmetrically normed real vector space. We call a pair of functions* $f = (f_1, f_2)$ *on* X *W-convex if for any* $x, y \in X$ *and* $\lambda \in [0, 1]$*, then*

$$f_j(W(x, y, \lambda)) \leq f_j(x) + (1 - \lambda)f_j(y)$$

for $j = 1, 2$.

Proposition

Suppose that $(X, \| \cdot |, W)$ is a convex asymmetrically normed real vector space. For all $t \in \mathbb{R}$ and $f = (f_1, f_2) \in \varepsilon(X, \| \cdot |)$,

(a) the pair of functions $f = (f_1, f_2)$ is $W-$ convex whenever W is translation invariant,

(b) the pair of functions $tf = ((tf)_1, (tf)_2)$ is $W-$ convex whenever W satisfies the homogeneity condition.

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Thank you for your attention $\ddot{\smile}$