On de Finetti's theorems

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based on joint work with V. Crismale and S. Rossi [a]

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[a] *De Finetti's theorems on quasi-local algebras and infinite Fermi tensor products* `to appear in IDAQP`

65th **Annual SAMS Congress**

06-08 December 2022

Stellenbosch University, South Africa

Outline

de Finetti's classical theorems

Statement

Let

- $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space
- $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables.

Definition

Such a sequence is said to be exchangeable if $\forall l \in \mathbb{N}$, $\forall \sigma \in \mathbb{P}_{\mathbb{Z}}$

$$\mathrm{P}(X_{n_1} \in A_1, X_{n_2} \in A_2, \dots, X_{n_l} \in A_l) = \mathrm{P}(X_{\sigma(n_1)} \in A_1, \dots, X_{\sigma(n_l)} \in A_l)$$

de Finetti's classical theorems

Statement

The investigation of distributional symmetries was initiated by B. de Finetti in 1931 with the study of 2-point valued random variables.

Theorem

If an infinite sequence of random variables $\{X_n\}$ *assuming only the values 0 and 1 is exchangeable, then there exists a unique probability measure* μ *on* $[0,1]$ *such that*

$$P(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) = \int_0^1 \theta^k (1-\theta)^{n-k} \mathrm{d}\mu(\theta)$$

de Finetti's classical theorems

A first classical generalization

This result has since found several generalizations.

Among these we mention the one made by Hewitt and Savage in 1955. The same statement continues to hold by considering the Tychonov product $X^{\mathbb{N}}$ of the compact Hausdorff space X instead of $\{0,1\}$ i.e.

Theorem

Symmetric (or exchangeable) probability measures on $X^{\mathbb{N}}$ are mixtures of product measures of the type $\mu \times \mu \times \cdots \times \mu \times \cdots$ where μ is a probability measure on X

Note that

$$C(X^{\mathbb{N}}) = \otimes_{\mathbb{N}} C(X) \Rightarrow X^{\mathbb{N}} = \sigma(\otimes_{\mathbb{N}} C(X))$$

Outline

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The noncommutative setting

Quantum stochastic processes

$\mathfrak{A}$ (unital) C^*-algebra, ω state on $\mathfrak{A}$ `Notation:` $\mathscr{S}(\mathfrak{A})$

One-to-one correspondence (see C. F. (2015), [3])

$\boxed{\text{quantum stochastic processes} \leftrightsquigarrow \text{state space } \mathscr{S}(*^{\mathbb{N}}\mathfrak{A})}$

More in detail

Definition

A quantum stochastic process is a quadruple $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathscr{H}, \xi)$:

- $\mathfrak{A}$ is a (unital) C^*-algebra
- $\mathscr{H}$ is a Hilbert space
- $\iota_j : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ is a $*$-representation for every $j \in \mathbb{N}$
- $\xi \in \mathscr{H}$ is a cyclic vector for the von Neumann algebra $\bigvee_{j \in \mathbb{N}} \iota_j(\mathfrak{A})$.

The noncommutative setting

Quantum stochastic processes

The bijection is realised as

$$\omega \in \mathscr{S}\left(*^{\mathbb{N}}\mathfrak{A}\right) \longleftrightarrow \iota_j := \pi_\omega \circ i_j$$

where

- $i_j : \mathfrak{A} \to *^{\mathbb{N}}\mathfrak{A}$: j-th embedding of $\mathfrak{A}$ into its infinite free product
- $(\mathscr{H}_\omega, \pi_\omega, \xi_\omega)$: GNS representation associated to the state ω

The noncommutative setting

Quantum stochastic processes

Recall also that

A process $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathscr{H}, \xi)$ is said to be exchangeable if $\forall n \in \mathbb{N}$, $\forall j_1 \neq j_2 \neq \cdots \neq j_n \in \mathbb{N}$, $\forall\, a_1, a_2, \ldots, a_n \in \mathfrak{A}$ and $\sigma \in \mathbb{P}_{\mathbb{N}}$ one has

$$\langle \iota_{j_1}(a_1)\iota_{j_2}(a_2)\cdots\iota_{j_n}(a_n)\xi, \xi\rangle = \langle \iota_{\sigma(j_1)}(a_1)\iota_{\sigma(j_2)}(a_2)\cdots\iota_{\sigma(j_n)}(a_n)\xi, \xi\rangle$$

Furthermore

- $\mathbb{P}_{\mathbb{N}}$ acts naturally on $*^{\mathbb{N}}\mathfrak{A}$: $\forall \sigma \in \mathbb{P}_{\mathbb{N}}\ \exists!\ \alpha_\sigma \in \mathrm{Aut}(*^{\mathbb{N}}\mathfrak{A})$ given by

$$\alpha_\sigma(i_{j_1}(a_1)i_{j_2}(a_2)\cdots i_{j_n}(a_n)) = i_{\sigma(j_1)}(a_1)i_{\sigma(j_2)}(a_2)\cdots i_{\sigma(j_n)}(a_n)$$

for $j_1 \neq j_2 \neq \cdots \neq j_n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in \mathfrak{A}$, $n \in \mathbb{N}$.

The noncommutative setting

Quantum stochastic processes

- A state ω on $\mathfrak{A}$ is α-invariant (symmetric) if $\omega \circ \alpha_\sigma = \omega$, for every $\sigma \in \mathbb{P}_\mathbb{N}$ $(\alpha : \mathbb{P}_\mathbb{N} \to \mathrm{Aut}(\mathfrak{A}))$

It is then clear that there is a bijection

$$\boxed{\text{exchangeable stochastic processes} \leftrightsquigarrow \text{symmetric states } \mathscr{S}(\ast^{\mathbb{N}}\mathfrak{A})}$$

`Notation:` $\mathscr{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$

The noncommutative setting

Stormer's version

In 1969 Størmer characterizes symmetric states of the minimal infinite tensor product:

Theorem

Let $\mathfrak{B} = \otimes^{\mathbb{N}} \mathfrak{A}$ the minimal infinite tensor product of an assigned C^*-algebra $\mathfrak{A}$. The convex set of symmetric states on $\mathfrak{B}$ is a Choquet simplex whose extreme points coincide with the closed set of infinite product states of a single state on $\mathfrak{A}$.

The Choquet simplex structure allows for a decomposition of any invariant state into an integral of extreme invariant states with respect to a unique probability measure.

Then, by virtue of the bijection above, a noncommutative version of the Hewitt and Savage theorem is established.

The non commutative setting

Infinite graded tensor product

Our focus: graded algebraic structures obtained as tensor products of $\mathbb{Z}_2$-graded $*$-algebras

(key role played in physics by the CAR algebra (C.F.2012, [4]).

By a $\mathbb{Z}_2$-graded C^*-algebra we mean a C^*-algebra that decomposes as $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{-1}$

(where $\mathbb{Z}_2 = \{1, -1\}$ with the product as the group operation), and involution and product given by:

$$(\mathfrak{A}_i)^* = (\mathfrak{A}^*)_i$$

$$\mathfrak{A}_i \mathfrak{A}_j \subset \mathfrak{A}_{ij} \quad i, j = 1, -1$$

Equivalently,

it is a pair $(\mathfrak{A}, \theta)$, where $\theta \in \mathrm{Aut}(\mathfrak{A})$ such that $\theta^2 = \mathrm{id}_{\mathfrak{A}}$

The noncommutative setting

Example: the CAR algebra

For an arbitrary set L, the

Canonical Anticommutation Relations

(CAR)

over L is the universal C^*-algebra generated by the set

$$\{a_j, a_j^\dagger \mid j \in L\}$$

and the relations

$$(a_j)^* = a_j^\dagger\,,\ \ \{a_j^\dagger, a_k\} = \delta_{jk}\mathbb{1}\,,$$

$$\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0\,,\ \ j, k \in L\,.$$

A $\mathbb{Z}_2$-grading is induced on $\mathrm{CAR}(\mathrm{L})$ by the $*$-automorphism θ acting on the generators as

$$\theta(a_j) = -a_j\,, \quad \theta(a_j^\dagger) = -a_j^\dagger\,, \quad j \in L\,.$$

The noncommutative setting

Our focus: infinite graded tensor products

$\mathfrak{A}_1$ and $\mathfrak{A}_2$ $\mathbb{Z}_2$-graded $*$-algebras, their algebraic tensor product

$$\mathfrak{A}_1 \odot \mathfrak{A}_2 = \oplus_{i,j\in\mathbb{Z}_2}(\mathfrak{A}_{1i} \odot \mathfrak{A}_{2j})$$

turns out to be an involutive $\mathbb{Z}_2$-graded algebra
which will be denoted by $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$ (Fermi tensor product)

with

$$x^* := \sum_{i,j\in\mathbb{Z}_2} \epsilon(i,j) x^*_{i,j}$$

$$xy := \sum_{i,j,k,l\in\mathbb{Z}_2} \epsilon(j,k) x_{i,j} \cdot y_{k,l}$$

for $x := \oplus_{i,j\in\mathbb{Z}_2} x_{i,j}$, $y := \oplus_{i,j\in\mathbb{Z}_2} y_{i,j} \in \oplus_{i,j\in\mathbb{Z}_2}(\mathfrak{A}_{1i} \odot \mathfrak{A}_{2j})$

where

$$\epsilon(i,j) := \begin{cases} -1 & \text{if } i = j = -1\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

The noncommutative setting

Our focus: infinite graded tensor products

$\boxed{\text{A linear functional } \omega : \mathfrak{A} \to \mathbb{C} \text{ is said to be even if } \omega \circ \theta = \omega}$

Notation: $\mathscr{S}_+(\mathfrak{A})$

$\mathfrak{A}_1, \mathfrak{A}_2$ $\mathbb{Z}_2$-graded C^*-algebras. Given $\omega_i \in \mathscr{S}(\mathfrak{A}_i)$, then the functional $\omega_1 \times \omega_2$ is defined as usual by

$$\omega_1 \times \omega_2 \left(\sum_{j=1}^{n} a_{1,j} \hat{\otimes} a_{2,j} \right) := \sum_{j=1}^{n} \omega_1(a_{1,j})\omega_2(a_{2,j})$$

for all $a = \sum_{j=1}^{n} a_{1,j} \hat{\otimes} a_{2,j} \in \mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$ It is positive on $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$ ($(\omega_1 \times \omega_2)(c^* c) \geq 0$) if at least one of them is even (A.M. (2003), [1]). Accordingly, one can consider its

GNS representation $\pi_{\omega_1 \times \omega_2}$

The noncommutative setting

Our focus: infinite graded tensor products

$$\pi_{\omega_i} : \mathfrak{A}_i \to \mathscr{B}(\mathscr{H}_{\omega_i}) \text{ GNS representations of } \omega_i \in \mathscr{S}_+(\mathfrak{A}_i)$$

$$\implies \pi_{\omega_1} \hat{\otimes} \pi_{\omega_2} \ \ast\text{-representation of } \mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 \text{ acting on } \mathscr{H}_{\omega_1} \hat{\otimes} \mathscr{H}_{\omega_2}$$

and

$$\pi_{\omega_1 \times \omega_2} \backsimeq \pi_{\omega_1} \hat{\otimes} \pi_{\omega_2}$$

By virtue of the latter, it makes sense to define the spatial norm

$$\boxed{\| \cdot \|_{\min} := \sup\{ \|\pi_\omega(\cdot)\| : \omega = \omega_1 \times \omega_2,\ \omega_i \in \mathscr{S}_+(\mathfrak{A}_i),\ i = 1, 2\}}$$

It is actually a norm (C.R.Z (2021), [5]).

More in general, one can consider the (minimal) infinite tensor Fermi product of a given C^*-algebra $\mathfrak{A}$ with itself

$$\hat{\otimes}^{\mathbb{N}}_{\min} \mathfrak{A}$$

The noncommutative setting

Our focus: infinite graded tensor products

If $\omega_i \in \mathscr{S}_+(\mathfrak{A}_i)$ are even states for every $i \in \mathbb{N}$ then their infinite product $\times_{i\in\mathbb{N}} \omega_i$, is the state on $\hat{\otimes}^{i\in\mathbb{N}}_{\min} \mathfrak{A}_i$ uniquely determined by

$$\times_{i\in\mathbb{N}} \omega_i(x_1 \hat{\otimes} \cdots \hat{\otimes} x_n \hat{\otimes} 1 \hat{\otimes} 1 \hat{\otimes} \cdots) = \omega_1(x_1)\omega_2(x_2)\cdots\omega_n(x_n)$$

for every $x_i \in \mathfrak{A}_i$, $i = 1, 2, \ldots, n$, and every $n \in \mathbb{N}$.

`Notation:` $\times^{\mathbb{N}}\omega$ product state when $\omega_i = \omega$, $\forall i \in \mathbb{N}$

This led us to a very precise characterization of the extreme symmetric states

The noncommutative setting

Our focus: infinite graded tensor products

Notation:

$\mathscr{E}(\mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\mathfrak{A}))$ set of the extreme symmetric states on $\mathfrak{A}$

Proposition (V. Crismale, S. Rossi, P.Z.)

If ω is a symmetric state on $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$, then the following are equivalent:

- $\omega \in \mathscr{E}(\mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))$
- there exists an even state $\rho \in \mathscr{S}(\mathfrak{A})$ such that $\omega = \times^{\mathbb{N}}\rho$

Furthermore

The set $\mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$ is still a Choquet simplex

The noncommutative setting

Our focus: infinite graded tensor products

Theorem

For any $\omega \in \mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$ there exists a unique probability measure μ supported on $\mathscr{E}(\mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))$ such that:

$$\omega = \int_{\mathscr{E}(\mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))} \psi \,\mathrm{d}\mu(\psi) = \int_{\mathscr{S}_{+}(\mathfrak{A})} \times^{\mathbb{N}} \rho \,\mathrm{d}\mu^{*}(\rho)$$

As a direct consequence we have

Proposition (V. Crismale, S. Rossi, P.Z.)

Any $\omega \in \mathscr{S}^{\mathbb{P}_{\mathbb{N}}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$ is even

References

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Thank you for your attention.