Asymptotics of the eigenvalues of fourth order Birkhoff regular boundary value problems

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Outline

1. Applications.
2. Introduction.
3. Self-adjoint fourth order problems
4. Birkhoff regular fourth order problems

Applications

Physical applications which are described by fourth order problems are vibration of strings and stability of flexible missiles.

Introduction

- Birkhoff has given an expansion theorem and an estimate for Green's function of the class of regular boundary conditions (Birkhoff regular), [Trans. Amer. Math. Soc. 9 (1908), 219-231].
- Salaff has done a deep investigation of Birkhoff regular boundary conditions and has provided a simplified version of these conditions. In addition, he has proved that self-adjointness implies Birkhoff regularity for even order operators, [Trans. Amer. Math. Soc. 134 (1968), N2, 355-373].
- Mennicken and Möller, using the boundary matrices, have given a simplified definition of Birkhoff regular problems, [*Non-self-adjoint boundary eigenvalue problems*, vol 192, Noth-Holland Publlishing Co., Amsterdam, 2003] .

Introduction

On the interval $[0, a]$ we consider the boundary value problem

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{1}$$

$$B_j(\lambda)y = 0,\ j = 1, 2, 3, 4, \tag{2}$$

where $a > 0$, $g \in C^1[0, a]$ is a real valued function.

The boundary terms $B_j(\lambda)y$ have the forms:

$$\begin{cases} y^{[p_1]}(0) = 0, \quad y^{[p_2]}(0) = 0, \\ y^{[p_3]}(a) + i\beta_3\lambda y^{[q_3]}(a) = 0, \quad y^{[p_4]}(a) + i\beta_4\lambda y^{[q_4]}(a) = 0, \end{cases} \tag{3}$$

with $0 \le p_1 < p_2 \le 3$, $0 \le q_3 < p_3 \le 3$, $0 \le q_4 < p_4 \le 3$ and $0 < p_3 < p_4 \le 3$.

We recall that the quasi-derivatives associated with (4) are given by

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = y'',\ y^{[3]} = y^{(3)} - gy',\ y^{[4]} = y^{(4)} - (gy')'.$$

Self-adjoint fourth order problems

On the interval $[0, a]$ we consider the boundary value problem

$$y^{(4)} - (gy')' = \lambda^2 y, \tag{4}$$

$$B_j(\lambda)y = 0,\ j = 1, 2, 3, 4, \tag{5}$$

The boundary terms $B_j(\lambda)y$ have the forms

- $B_j(\lambda)y = y^{[p_j]}(a_j)$ or
- $B_j(\lambda)y = y^{[p_j]}(a_j) + i\varepsilon_j\alpha\lambda y^{[q_j]}(a_j)$,

where $a_j = 0$ for $j = 1, 2$, $a_j = a$ for $j = 3, 4$, $\alpha > 0$ and $\varepsilon_j \in \{-1, 1\}$.

Self-adjoint fourth order problems

Define

$$\Theta_1 = \{s \in \{1,2,3,4\} : B_s(\lambda) \text{ depends on } \lambda\}, \ \Theta_0 = \{1,2,3,4\} \backslash \Theta_1,$$
$$\Theta_1^0 = \Theta_1 \cap \{1,2\}, \quad \Theta_1^a = \Theta_1 \cap \{3,4\}.$$

Assumption

The numbers p_1, p_2, q_j for $j \in \Theta_1^0$ are distinct as well as the numbers p_3, p_4, q_j for $j \in \Theta_1^a$.

Self-adjoint fourth order problems

We denote by U the collection of the boundary conditions (5) and define the following operators related to U:

$$U_0 y = (y^{[p_j]}(a_j))_{j\in\Theta_1} \text{ and } U_1 y = (\varepsilon_j y^{[q_j]}(a_j))_{j\in\Theta_1}, \quad y \in W_4^2(0,a).$$

We put $k = |\Theta_1|$ and consider the linear operators $A(U)$, K and M in the space $L_2(0,a) \oplus \mathbb{C}^k$ with domains

$$\mathscr{D}(A(U)) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ U_1 y \end{pmatrix} : \ y \in W_4^2(0,a), \ y^{[p_j]}(a_j) = 0 \text{ for } j \in \Theta_0 \right\},$$

$\mathscr{D}(K) = \mathscr{D}(M) = L_2(0,a) \oplus \mathbb{C}^k$ given by

$$(A(U))\widetilde{y} = \begin{pmatrix} y^{[4]} \\ U_0 y \end{pmatrix} \text{ for } \widetilde{y} \in \mathscr{D}(A(U)), \ K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Self-adjoint fourth order problems

The conditions under which the differential operator $A(U)$ is self-adjoint are given by

Theorem (Möller & Zinsou, Theorem 1.2, QM, 34(2011))

Denote by P_0 the set of p in $y^{[p]}(0) = 0$ for the λ-independent boundary conditions and by P_a the corresponding set for $y^{[p]}(a) = 0$. Then the differential operator $A(U)$ associated with this boundary value problem is self-adjoint if and only if $p + q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\varepsilon_j\lambda y^{[q]}(a_j) = 0$ and $\varepsilon_j = 1$ if q is even in case $a_j = 0$ or odd in case $a_j = a$, $\varepsilon_j = -1$ otherwise, $\{0,3\} \not\subset P_0$, $\{1,2\} \not\subset P_0$, $\{0,3\} \not\subset P_a$ and $\{1,2\} \not\subset P_a$.

Self-adjoint fourth order problems

We will distinguish the following different cases of boundary conditions at the endpoint 0:

$$\begin{cases} \text{Case 1} : (\mathrm{p}_1, \mathrm{p}_2) = (0, 1), \ \text{Case 2} : (\mathrm{p}_1, \mathrm{p}_2) = (0, 2), \\ \text{Case 3} : (\mathrm{p}_1, \mathrm{p}_2) = (1, 3), \ \text{Case 4} : (\mathrm{p}_1, \mathrm{p}_2) = (2, 3). \end{cases} \tag{6}$$

Hence, we get

$$y^{[p_j]}(0) = 0, j = 1, 2 \tag{7}$$

$$y^{[p_j]}(a) + i\alpha\varepsilon_j\lambda y^{[q_j]}(a) = 0, j = 3, 4, \tag{8}$$

$0 \leq p_1 < p_2 \leq 3$, $p_1 + p_2 \neq 3$, $(p_3, q_3) = (2, 1)$, $(p_4, q_4) = (3, 0)$, $\varepsilon_3 = 1$ and $\varepsilon_4 = -1$.

Self-adjoint fourth order problems

The asymptotics of the eigenvalues are given by

Theorem (Möller & Zinsou, Theorem 5.2, Complex Anal. Oper. Theory , **6**(2012))

For $g \in C^1[0,a]$*, there exists a positive integer* k_0 *such that the eigenvalues* λ_k*,* $k \in \mathbb{Z}$ *of the problem* (4)*,* (7)*,* (8)*, where* $B_1(y) = y^{[p_1]}(0)$*,* $B_2(y) = y^{[p_2]}(0)$*,* $B_3y = y''(a) + \lambda y'(a)$ *and* $B_4y = y^{[3]}(a) - i\lambda y''(a)$ *are* $\lambda_{-k} = -\overline{\lambda_k}$*,* $\lambda_k = \mu_k^2$ *for* $k \geq k_0$ *and the* μ_k *have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})$$

and the numbers τ_0*,* τ_1*,* τ_2 *are as follows:*

Self-adjoint fourth order problems

Theorem (cont)

Case 1: $p_1 = 0, p_2 = 1,$

$$\tau_0 = -\frac{\pi}{4a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{i}{2}\,\frac{1+\alpha^2}{\pi\alpha},$$
$$\tau_2 = \frac{1}{16}\,\frac{G(a)}{\pi} + \frac{i}{8}\,\frac{1+\alpha^2}{\pi\alpha} - \frac{a((1-\alpha^2)^2 + \alpha^2 g(0))}{4\pi^2 a^2}.$$

Case 2: $p_1 = 0, p_2 = 2,$

$$\tau_0 = -\frac{\pi}{2a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{i}{2}\,\frac{1+\alpha^2}{\pi\alpha},$$
$$\tau_2 = \frac{1}{8}\,\frac{G(a)}{\pi} + \frac{i}{4}\,\frac{1+\alpha^2}{\pi\alpha} - \frac{a(1-\alpha^2)^2)}{4\pi^2 a^2}.$$

Self-adjoint fourth order problems

Theorem (cont)

Case 3: $p_1 = 1, p_2 = 3,$

$$\tau_0 = -\frac{\pi}{a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{i}{2}\,\frac{1+\alpha^2}{\pi\alpha},$$
$$\tau_2 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{i}{2}\,\frac{1+\alpha^2}{\pi\alpha} - \frac{a(1-\alpha^2)^2}{4\pi^2 a^2}.$$

Case 4: $p_1 = 2, p_2 = 3,$

$$\tau_0 = -\frac{5\pi}{4a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{i}{2}\,\frac{1+\alpha^2}{\pi\alpha},$$
$$\tau_2 = \frac{5}{16}\,\frac{G(a)}{\pi} + \frac{5i}{8}\,\frac{1+\alpha^2}{\pi\alpha} - \frac{a((1-\alpha^2)^2 - 3\alpha^2 g(0))}{4\pi^2 a^2}.$$

In particular, there is an even number of the pure imaginary eigenvalues in each case.

Birkhoff regular fourth order problems

Questions

1. *What happens if we drop the self-adjointness conditions?*
2. *Do we still obtain the same expansions?*

We need to choose our boundary conditions such the problem is Birkhoff regular.

We define

$$\Theta_1 = \{s \in \{1,2,3,4\} : B_s(\lambda) \text{ depends on } \lambda\}\,,\ \Theta_0 = \{1,2,3,4\} \backslash \Theta_1, \tag{9}$$

$$\Theta_1^0 = \Theta_1 \cap \{1,2\}, \quad \Theta_1^a = \Theta_1 \cap \{3,4\}, \tag{10}$$

and

$$\Lambda = \{s \in \{1,2,3,4\} : p_s > -\infty\}\,,\ \Lambda^0 = \Lambda \cap \{1,2\},\ \Lambda^a = \Lambda \cap \{3,4\}. \tag{11}$$

Birkhoff regular fourth order problems

Assumption

We assume that the numbers p_s *for* $s \in \Lambda^0$, q_j *for* $j \in \Theta_1^0$ *are distinct and that the numbers* p_s *for* $s \in \Lambda^a$, q_j *for* $j \in \Theta_1^a$ *are distinct.*

Let $p_j, q_j \in \{0, 1, 2, 3\}$, where p_j, q_j are as defined in Assumption 0.3, $j = 1, 2, 3, 4$. Let u such that $u = 0$ if $j = 1, 2$ and $u = 1$ if $j = 3, 4$. Let $C(r, u)$, $r = 1, 2, 3, 4, 5$, $u = 0, 1$, be the following conditions, for $l = 1, 2$:

$C(1, u)$: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} > q_{2+2u} + 2$;
$C(2, u)$: $p_{1+2u} > q_{1+2u} + 2$, $q_{2+2u} + 2 > p_{2+2u}$;
$C(3, u)$: $p_{1+2u} > q_{1+2u} + 2$, $p_{2+2u} = q_{2+2u} + 2$ and $\beta_{2+2u} \neq (-1)^l$;
$C(4, u)$: $q_{1+2u} + 2 > p_{1+2u}$, $q_{2+2u} + 2 > p_{2+2u}$;
$C(5, u)$: $q_{1+2u} + 2 > p_{1+2u}$, $p_{2+2u} = q_{2+2u} + 2$,

$$\beta_{2+2u} \neq \begin{cases} (-1)^{l+1} & \text{if} \quad q_{1+2u} - q_{2+2u} = 1, \\ (-1)^l & \text{if} \quad q_{1+2u} - q_{2+2u} = 3. \end{cases}$$

Birkhoff regular fourth order problems

Using the classifications done by Zinsou: Fourth order Birkhoff regular problems with eigenvalue parameter dependent boundary conditions. Turk J Math **40** (2016), 864–873, we have the following proposition

Proposition

The problem (4), (5) *is Birkhoff regular if and only if there are* $r_0, r_1 \in \{1, 2, 3, 4, 5\}$ *such that the conditions* $C(r_0, 0)$ *and* $C(r_1, 1)$ *hold.*

Hence, we write the boundary conditions (5) as

$$\begin{cases} y^{[p_1]}(0) = 0, \quad y^{[p_2]}(0) = 0, \\ y^{[p_3]}(a) + i\beta_3\lambda y^{[q_3]}(a) = 0, \quad y^{[p_4]}(a) + i\beta_4\lambda y^{[q_4]}(a) = 0, \end{cases} \tag{12}$$

where $0 \leq p_1 < p_2 \leq 3$, $0 \leq q_3 < p_3 \leq 3$, $0 \leq q_4 < p_4 \leq 3$ and $0 < p_3 < p_4 \leq 3$.

Birkhoff regular fourth order problems

We will distinguish the following different cases of boundary conditions at the endpoint 0:

$$\begin{cases} \text{Case 1} : (\mathrm{p}_1, \mathrm{p}_2) = (0, 1), \ \text{Case 2} : (\mathrm{p}_1, \mathrm{p}_2) = (0, 2), \\ \text{Case 3} : (\mathrm{p}_1, \mathrm{p}_2) = (0, 3), \ \text{Case 4} : (\mathrm{p}_1, \mathrm{p}_2) = (1, 2), \\ \text{Case 5} : (\mathrm{p}_1, \mathrm{p}_2) = (1, 3), \ \text{Case 6} : (\mathrm{p}_1, \mathrm{p}_2) = (2, 3). \end{cases} \tag{13}$$

However the boundary conditions at the right endpoint a will be classified as

$$y''(a) + i\beta_3\lambda y, (a) = 0, \quad y^{[3]}(a) + i\beta_4\lambda y(a) = 0. \tag{14}$$

Birkhoff regular fourth order problems

Theorem

For $g \in C^1[0,a]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$ of the problem (4), (12), *where* $B_1(y) = y^{[p_1]}(0)$, $B_2(y) = y^{[p_2]}(0)$, $B_3y = y''(a) + i\beta_3\lambda y'(a)$ *and* $B_4y = y^{[3]}(a) + i\beta_4\lambda y(a)$ *are* $\lambda_{-k} = -\overline{\lambda_k}$, $\lambda_k = \mu_k^2$ *for* $k \geq k_0$ *and the* μ_k *have the asymptotics*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})$$

and the numbers τ_0, τ_1, τ_2 *are as follows:*

Case 1: $p_1 = 0$, $p_2 = 1$,

$$\tau_0 = -\frac{\pi}{4a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{1}{2}\,\frac{i}{\pi}\,\frac{1-\beta_3\beta_4}{\beta_3},$$
$$\tau_2 = \frac{1}{16}\,\frac{G(a)}{\pi} - \frac{1}{4}\,\frac{ag(0)}{\pi^2} - \frac{1}{4}\,\frac{a(\beta_3^2\beta_4^2 + 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} + \frac{1}{8}\,\frac{i}{\pi}\,\frac{1-\beta_3\beta_4}{\beta_3}.$$

Birkhoff regular fourth order problems

Theorem (cont)

Case 2: $p_1 = 0, p_2 = 2$

$$\tau_0 = -\frac{\pi}{2a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{1}{2}\,\frac{i}{\pi}\,\frac{1-\beta_3\beta_4}{\beta_3},$$

$$\tau_2 = \frac{1}{8}\,\frac{G(a)}{\pi} - \frac{1}{4}\,\frac{a(\beta_3^2\beta_4^2 + 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} + \frac{1}{4}\,\frac{i}{\pi}\,\frac{1-\beta_3\beta_4}{\beta_3}.$$

Case 3: $p_1 = 0, p_2 = 3,$

$$\tau_0 = -\frac{5\pi}{4a},\ \tau_1 = \frac{1}{2}\,\frac{\beta_3\beta_4 - 1}{\pi\beta_3} - \frac{i}{4}\,\frac{G(a)}{\pi},$$

$$\tau_2 = -\frac{1}{16}\,\frac{aG^2(a)}{\pi^2} + \frac{1}{4}\,\frac{aG(a)}{\pi^2}\,\frac{\beta_4\beta_3 - 1}{\beta_3} + \frac{5}{8}\,\frac{\beta_4\beta_3 - 1}{\pi\beta_3} - \frac{1}{4}\,\frac{a(\beta_3^2\beta_4^2 - 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} - \frac{5i}{16}\,\frac{G(a)}{\pi}.$$

Birkhoff regular fourth order problems

Theorem (cont)

Case 4: $p_1 = 1, p_2 = 2,$

$$\tau_0 = -\frac{5\pi}{4a},\ \tau_1 = \frac{1}{2}\,\frac{\beta_3\beta_4 - 1}{\pi\beta_3} - \frac{i}{4}\,\frac{G(a)}{\pi},$$

$$\tau_2 = -\frac{1}{16}\,\frac{aG^2(a)}{\pi^2} + \frac{1}{4}\,\frac{aG(a)}{\pi^2}\,\frac{\beta_4\beta_3 - 1}{\beta_3} + \frac{5}{8}\,\frac{\beta_4\beta_3 - 1}{\pi\beta_3} - \frac{1}{4}\,\frac{a(\beta_3^2\beta_4^2 - 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} - \frac{5i}{16}\,\frac{G(a)}{\pi}.$$

Case 5: $p_1 = 1, p_2 = 3,$

$$\tau_0 = -\frac{\pi}{a},\ \tau_1 = \frac{1}{4}\,\frac{G(a)}{\pi} + \frac{1}{2}\,\frac{i}{\pi}\,\frac{1 - \beta_3\beta_4}{\beta_3},$$

$$\tau_2 = \frac{1}{4}\,\frac{G(a)}{\pi} - \frac{1}{4}\,\frac{a(\beta_3^2\beta_4^2 + 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} - \frac{1}{2}\,\frac{i}{\pi}\,\frac{1 - \beta_3\beta_4}{\beta_3}.$$

Birkhoff regular fourth order problems

Theorem (cont)

Case 6: $p_1 = 2$, $p_2 = 3$,

$$\tau_0 = -\frac{5\pi}{4a},\ \tau_1 = \frac{1}{4}\ \frac{G(a)}{\pi} + \frac{1}{2}\ \frac{i}{\pi}\ \frac{1-\beta_3\beta_4}{\pi\beta_3},$$

$$\tau_2 = \frac{5}{16}\ \frac{G(a)}{\pi} + \frac{3}{4}\ \frac{ag(0)}{\pi^2} - \frac{1}{4}\ \frac{a(\beta_3^2\beta_4^2 + 2\beta_3\beta_4 + 1)}{\pi^2\beta_3^2} + \frac{5}{8}\ \frac{i}{\pi}\ \frac{1-\beta_3\beta_4}{\pi\beta_3}.$$

In particular, there is an odd number of pure imaginary eigenvalues in each case.

Thank you