Blow-up in a nonlinear heat equation: before, at, and after

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forward together
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Nonlinear heat equations:
Introduction

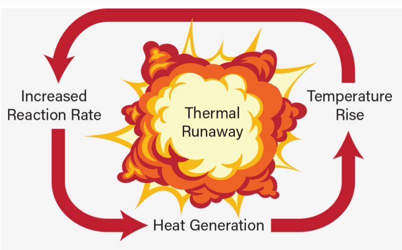
Consider nonlinear heat equations of the form

$$u_t = u_{xx} + f(u), \qquad f(u) = \begin{cases} u^2 & \leftarrow \text{considered here} \\ u^m, & m = 3, 4, 5, \ldots \\ e^u, & \text{etc.} \end{cases}$$

- The linear term u_{xx} models diffusion (energy loss)
- The nonlinear term $f(u)$ models a reaction process (energy gain)

Applications:

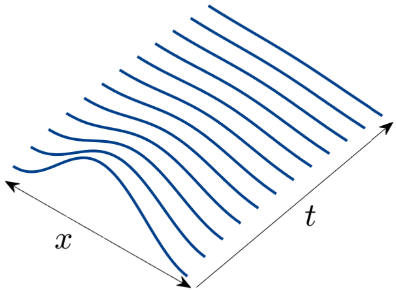
- chemistry (thermal runaway)
- fluids (singularity formation)
- biology (bacterial communication)
- etc.



Nonlinear heat equations:
A competition between smoothing and focusing

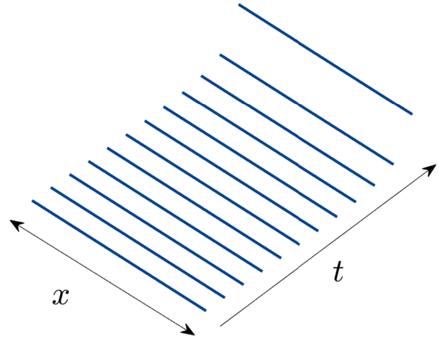
$$\boxed{u_t = u_{xx} + u^2}$$

Only the linear term: $u_t = u_{xx}$



Solutions: $u = e^{-t}\cos x, e^{-t}\sin x$ etc.

Only the nonlinear term: $u_t = u^2$

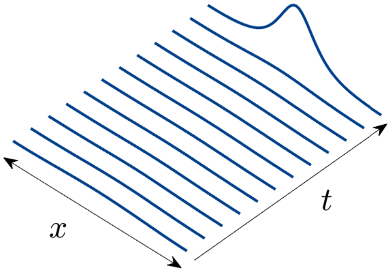


Solution: $u = u_0/(1 - u_0 t)$

Nonlinear heat equations:

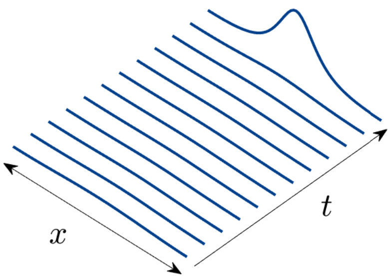
Steady state or blow-up?

Now both terms: $u_t = u_{xx} + u^2$



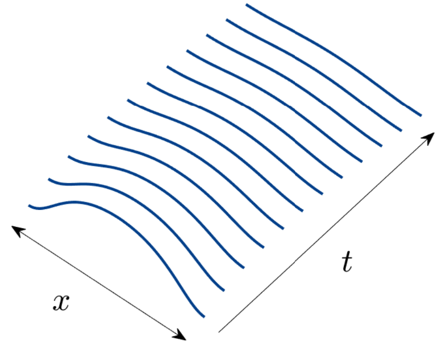
Blow-up in finite time

Now both terms: $u_t = u_{xx} + u^2$



Blow-up in finite time

But if initial condition is strictly negative



a steady state solution is also possible

Outline of talk:

Numerics, asymptotics, and speculation

- Numerical computation of blow-up solutions:
  - Adaptive rescaling, moving meshes, etc (Berger & Kohn (1988), Budd et. al (1996))
  - Reciprocal substitution (Keller & Lowengrub (1993))
  - Fourier spectral methods
- Before blow up:
  - Simple perturbation analysis
- At blow up:
  - Matched asymptotic expansions
- After blow up: $(*) \longrightarrow$
  - Numerical exploration
- The complex viewpoint:
  - Numerical analytic continuation

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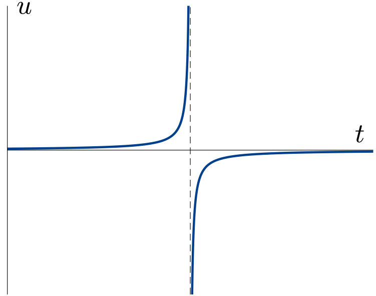
$(*)$ Solution of ODE exists after blow up. What about PDE?

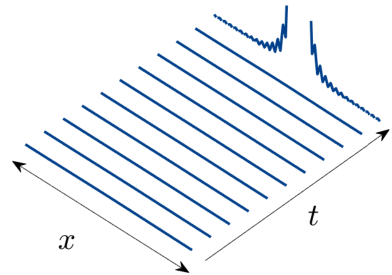
$$u_t = u^2 \quad \Rightarrow \quad u = \frac{1}{(1/u_0) - t}$$



Numerical computation of blow-up solutions:

Possible approaches

Standard off-the-shelf methods inevitably lead to disaster as shown alongside $\longrightarrow$



As long as solutions remain positive the best strategy is to set $u = 1/v$, then

$$u_t = u_{xx} + u^2 \quad \Rightarrow \quad v_t = v_{xx} - 1 - 2\frac{(v_x)^2}{v}$$

Better for numerics, since v stays bounded

Solve v-equation with Fourier spectral method:

$$v(x,t) = \sum_{k=-\infty}^{\infty} c_k(t) e^{ikx} \quad \Rightarrow \quad \frac{dc_k}{dt} = -k^2 c_k - \delta_k - d_k, \quad |k| \le K$$

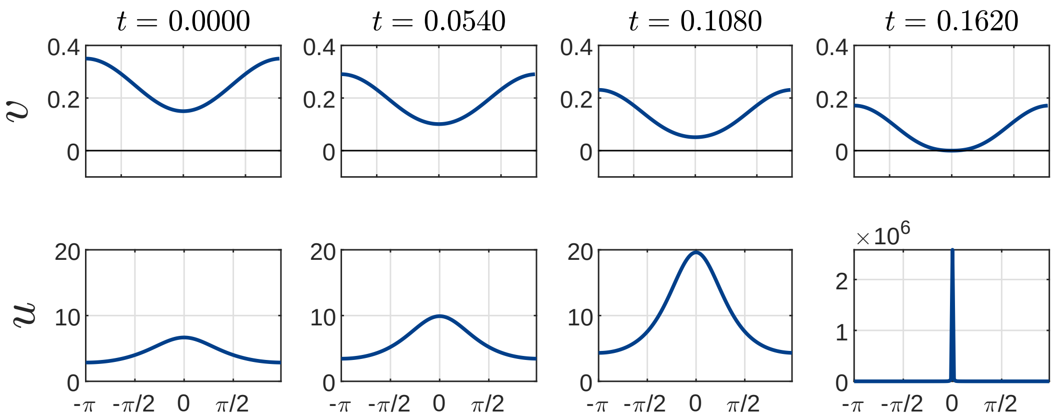
Integrate the system on the right with an ODE solver that has error control

Numerical computation of blow-up solutions:

Success of the reciprocal substitution

Consider initial conditions of the form (periodic, positive, single local max/min)

$$u(x,0) = \frac{1}{\alpha - \epsilon \cos x}, \quad v(x,0) = \alpha - \epsilon \cos x, \quad 0 < \epsilon < \alpha \quad \text{(below } \alpha = 0.25,\ \epsilon = 0.1\text{)}$$



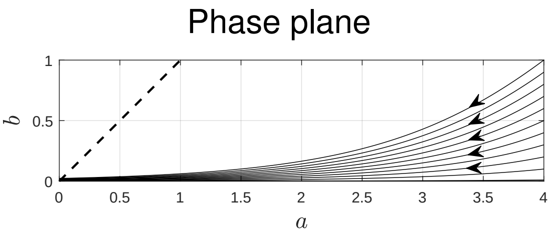
Before blow up:

A two-mode approximation

Key observation: v-solution is smooth, approximate by truncated Fourier series

$$v \approx a(t) - b(t)\cos x, \quad a(0) = \alpha, \ b(0) = \epsilon$$

Plug into v-equation, drop $\cos 2x$ terms



$$a\frac{da}{dt} + \frac{1}{2}b\frac{db}{dt} = -a - \frac{3}{2}b^2, \quad b\frac{da}{dt} + a\frac{db}{dt} = -ab - b$$

No explicit solution, but amenable to a perturbation analysis when $0 < \epsilon \ll \alpha$:

$$a\frac{da}{dt} + \cancel{\frac{1}{2}b\frac{db}{dt}} = -a - \cancel{\frac{3}{2}b^2} \quad \Rightarrow \quad \frac{da}{dt} = -1 \quad \Rightarrow \quad a = \alpha - t$$

$$\cancel{b\frac{da}{dt}} + a\frac{db}{dt} = -ab - \cancel{b} \quad \Rightarrow \quad \frac{db}{dt} = -b \quad \Rightarrow \quad b = \epsilon e^{-t}$$

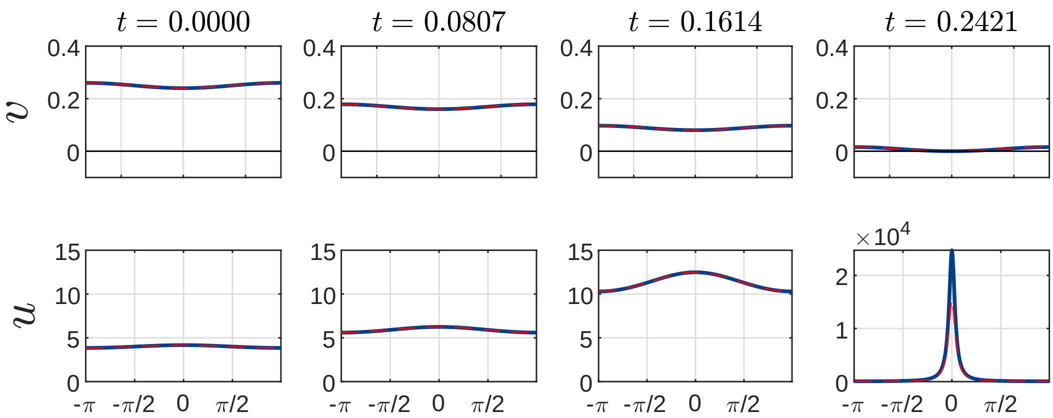
Before blow up:

Accuracy of perturbation solution

As a check on the approximation just derived, plug it into PDE:

$$V = \alpha - t - \epsilon\, e^{-t}\cos x \quad \Rightarrow \quad V\Big(V_t - V_{xx} + 1 + 2(V_x)^2/V\Big) = 2\,\epsilon^2 e^{-2t}\sin^2 x$$

Parameters: $\alpha = 0.25$, $\epsilon = 0.01$, $t_c \approx 0.2421$



At blow up:

Method of matched asymptotic expansions

Previous perturbation analysis no longer valid since $0 < b \ll a$ is violated.

A modified approach gives the profile at the critical time as

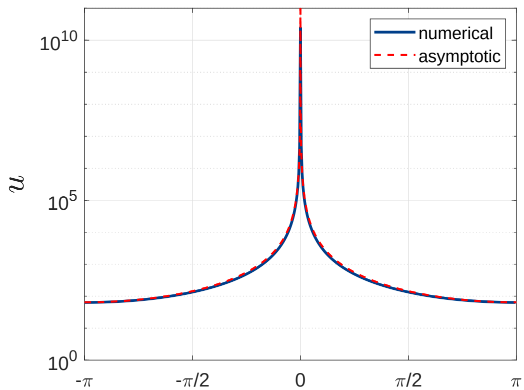
$$v \sim 2\epsilon\, e^{-\alpha} \sin^2(x/2) + 2\epsilon^2 \sin^2 x \Big(e^{-2\alpha} \log\Big(2\epsilon\, e^{-\alpha} \sin^2(x/2)\Big) + C_1(\alpha) + C_2(\alpha)\Big)$$

Shown alongside is the corresponding u-solution at the critical time. (Note log-scale.) Same parameters as before:

$$\alpha = 0.25,\ \epsilon = 0.01,\ t_c \approx 0.2421$$

Modification necessary when $|x| \ll 1$:

$$v \sim \frac{-x^2}{16 \log |x|}$$



After blow up:

A classic paper from the 1990s

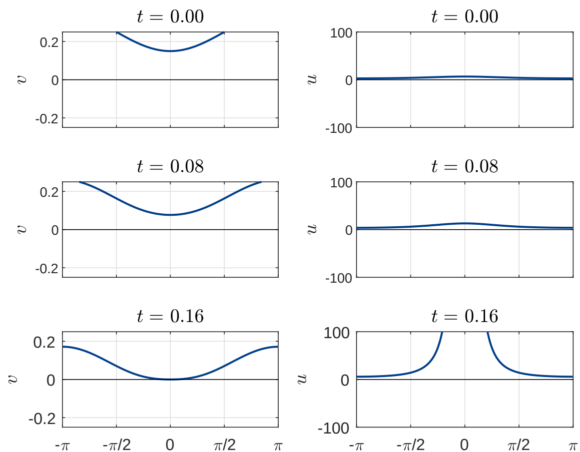
J.B. Keller & J.S. Lowengrub. “Asymptotic and numerical results for blowing-up solutions to semilinear heat equations” (1993)

- Derived estimates for blow-up time and solution profile at blow-up
- Suggested the reciprocal substitution for numerical computations
- Used a finite difference method to solve the v-equivalent of $u_t = u_{xx} + u^5$
- Concluded the following

> “Thus, the calculation actually continues the solution slightly beyond the blow-up time of the original solution u. The method becomes unstable a short time after the blow-up happens, however.”
>
> – Keller & Lowengrub, 1992

After blow up:
Keller & Lowengrub revisited

v-solution u-solution

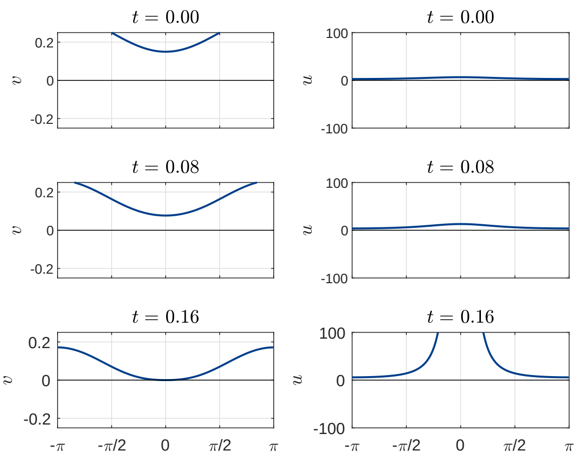


What happens next?

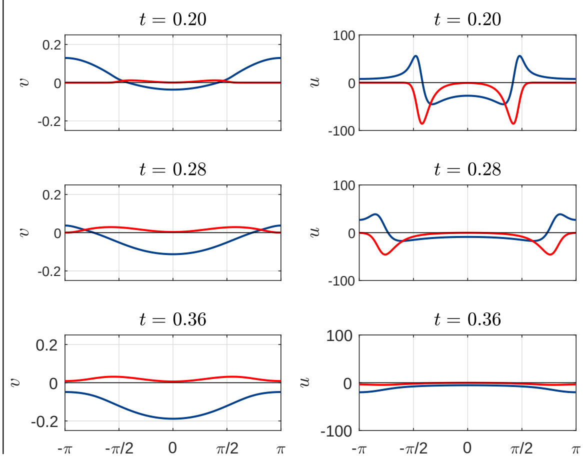
After blow up:

Keller & Lowengrub revisited

v-solution u-solution



The solution turns complex



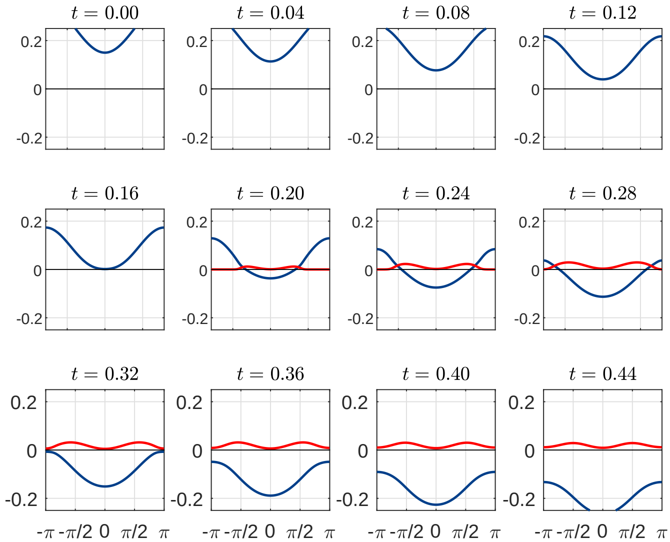
After blow up:

Dynamics of the v-solution

v-solution

$$\alpha = 0.25$$
$$\epsilon = 0.1$$

Not unique!



Blue is real part

Red imaginary

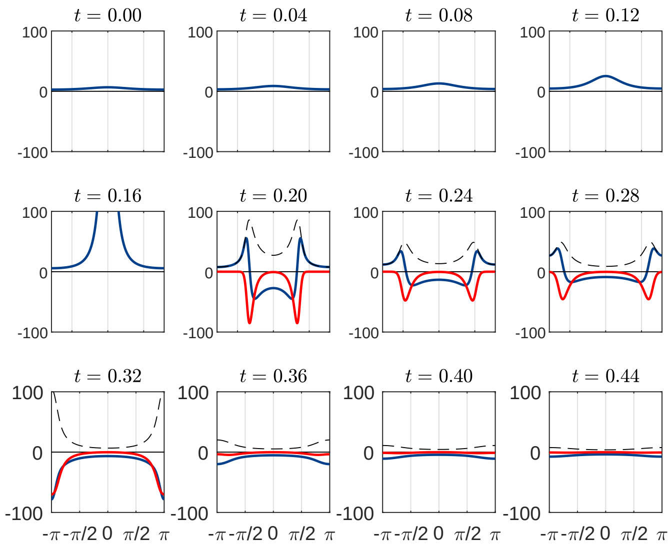
After blow up:

Dynamics of the u-solution

u-solution

$$\alpha = 0.25$$
$$\epsilon = 0.1$$

Note secondary 'blow up' $\longrightarrow$ (almost)



Blue is real part

Red imaginary

After blow up:

What is known about post blow-up solutions?

Theory:

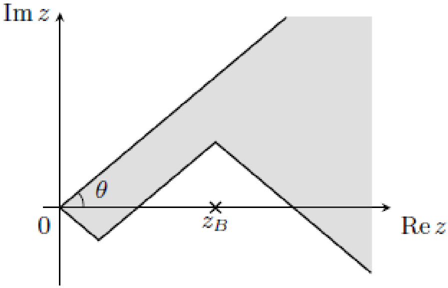
K. Masuda. "Analytic solutions of some nonlinear diffusion equations" (1984)

- Possibility of complex solutions after critical time, uniqueness may be lost

Numerics:

C.-H. Cho et al. "A blow-up problem for a nonlinear heat equation in the complex plane of time" (2016)

A. Takayashu et al. "Rigorous numerics for nonlinear heat equations in the complex plane of time" (2020)



- Both numerical papers consider methods based on the complexification of t

After blow up:

How to compute through singularity?

Main point: the computational setup should allow for complex solutions

1. Fourier spectral method, based on $v(x,t) = \sum_{k=-\infty}^{\infty} c_k(t) e^{ikx}$

Complex values introduced at roundoff level because of complex Fourier series

2. Alternative approach: split PDE into its real and imaginary parts $v = p + iq$

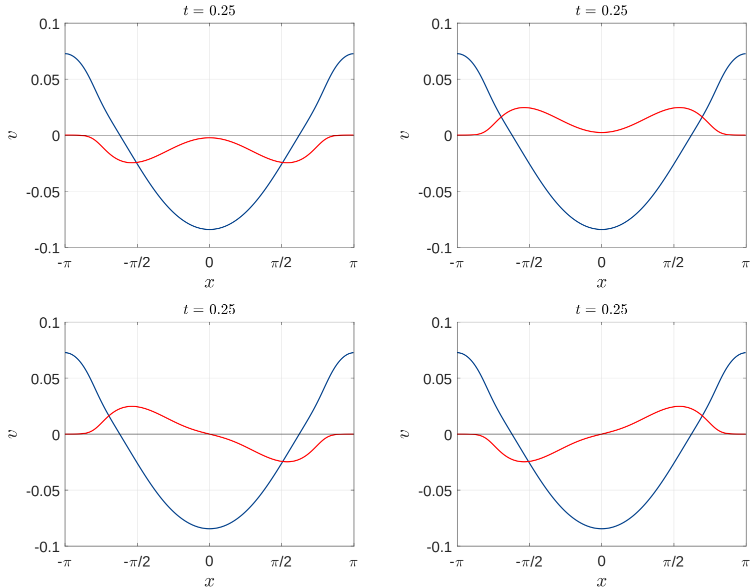
$$v_t = v_{xx} - 1 - 2\frac{(v_x)^2}{v} \quad \Rightarrow \quad \begin{aligned} p_t &= f(p,q) \\ q_t &= g(p,q) \end{aligned}$$

Initialize q with random initial perturbation at machine roundoff level
Solve with finite differences, pseudospectral methods, etc

> Conjecture: Keller & Lowengrub failed because they solved the v-equation as a real equation using real arithmetic

Numerical computation of blow-up solutions:

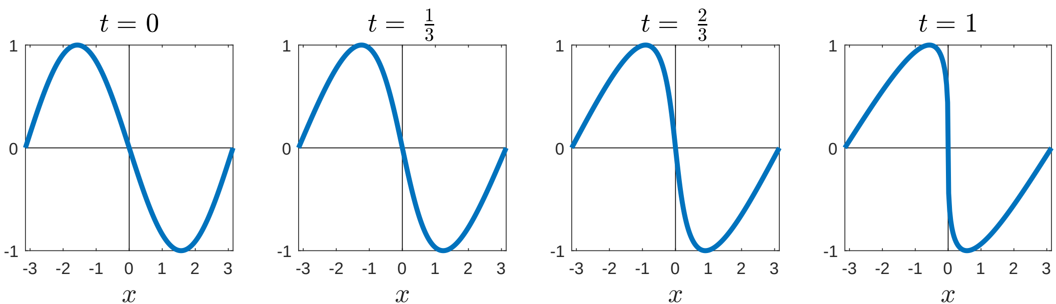
Solutions observed after blow up



> "The shortest path between two truths in the real domain often goes through the complex plane" – Paul Painlevé (1863–1933)

Consider a different nonlinear PDE that exhibits blow up (in gradient):

Inviscid Burgers: $u_t + uu_x = 0, \quad u(x,0) = -\sin x$



The view from the complex x-plane:

A classic paper from the 1980s

Pole condensation and the Riemann surface associated with a shock in Burgers' equation

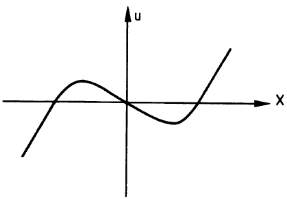
D. Bessis

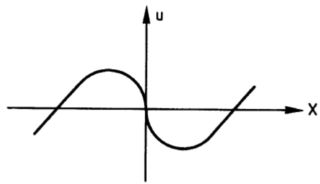
Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

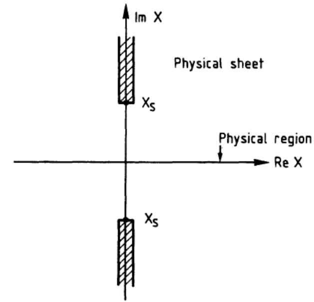
and J. D. Fournier

Observatoire, Mont-Gros, BP 139, 06003 Nice Cedex, France

J. Physique Lett., Vol. 45 (1984)







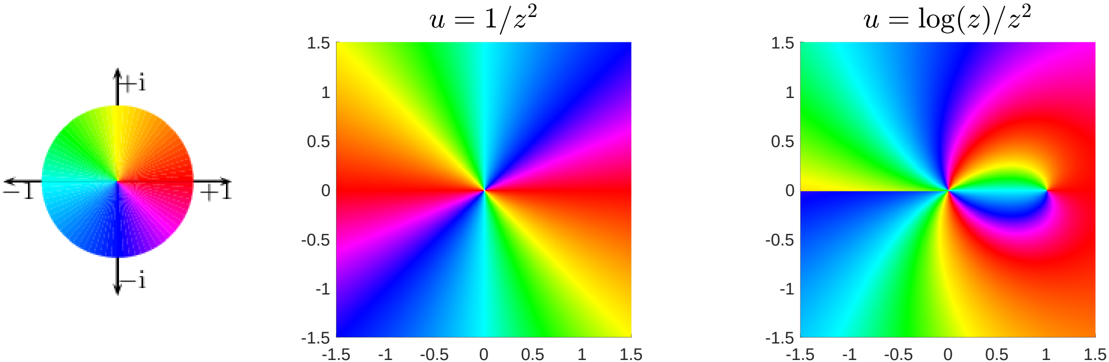
"Starting from $t = 0$ at infinity, the moving square root singularities x_s and $-x_s$ come down along the imaginary axis; they meet at $t = t_*$ and they move away along the real axis." – Bessis & Fournier (1984)

The view from the complex x-plane:

How to compute singularity structure in the absence of an explicit solution?

Procedure suggested in SIADS (2003)

- Solve PDE on real line using a Fourier spectral method
- At any time t, extend the solution from the real line into the complex plane using numerical analytic continutation (such as Fourier-Padé methods)
- Display the continued solution using a phase plot: $u = re^{i\theta}$



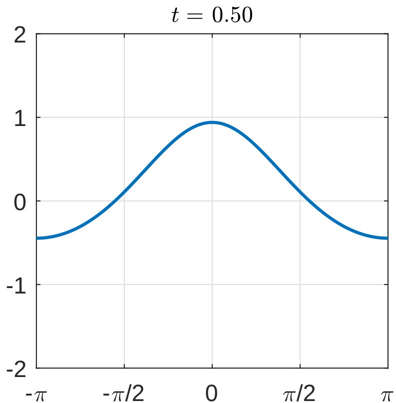
The view from the complex x-plane:

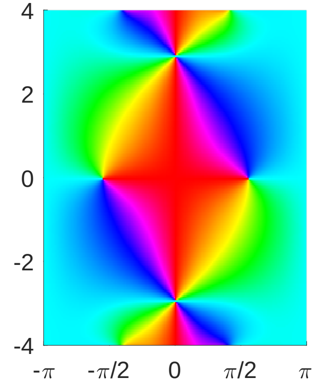
Nonlinear heat equation at small times

$$u(x,0) = \cos x$$

At $t = 0$, no singularities in finite complex plane

At $t > 0$ singularities start to move in from infinity along imaginary axis





Looks like poles of order 2 but analysis says that locally, near singularity

$$u = \frac{c_1}{\zeta^2} + \frac{c_2}{\zeta} + c_3 + c_4\zeta + c_5\zeta^2 + c_6\zeta^3 + c_6\zeta^4 + c_7\zeta^4 \log\zeta + \cdots$$