# The Jacobson Property in Banach algebras

A Swartz, H Raubenheimer

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In this talk we discuss subsets of A that have the Jacobson Property, i.e.  $X \subseteq A$  such that for  $a, b \in A$ :

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If  $X \subseteq A$  we say that X has the Jacobson Property, if for  $a, b \in A$ :

$$1 - ab \in X \implies 1 - ba \in X$$
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An ideal *J* in *A* is said to be *inessential* 

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 $A_{ai}$  - the set of almost invertible elements in A.

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they are those elements which can be written as a product of an invertible and an idempotent:

$$A^{-1}A^{\bullet} = A^{\bullet}A^{-1} = \{a \in A : a \in aA^{-1}a\} \subseteq \widehat{A}.$$
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## Definition

Let *I* be an ideal in a Banach algebra *A*. *I* is *subprime* if  $ab \in I \implies a \in I$  or  $b \in I$ .



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The notion of a subprime ideal defined above coincides with that of a *prime* ideal in a commutative ring or algebra.

Also called completely prime.



From V. Müller - 1990's:

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- (i) if  $a \in A$ ,  $n \in \mathbb{N}$  then  $a \in R \iff a^n \in R$ ;
- (ii) if a, b, c, d are mutually commuting elements of A satisfying ac + bd = 1 then  $a, b \in R \iff ab \in R$ .

## **Theorem**

Let R be a nonempty subset of a Banach algebra A satisfying:

$$ab \in R \iff a, b \in R$$
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for all commuting elements  $a, b \in A$ . Then R is a regularity.

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The definition of a regularity can be divided into two parts as follows:

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A nonempty subset *R* in a Banach algebra *A* will be called a *lower semiregularity* if it satisfies the following conditions:



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## Definition

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- (i)  $a \in A$ ,  $n \in \mathbb{N}$  and  $a^n \in R \Rightarrow a \in R$ ,
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A nonempty subset *R* in a Banach algebra *A* is called an *upper semiregularity* if it satisfies the following conditions:

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any semigroup containing a neighbourhood of the identity is an upper semiregularity,



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Let R be a nonempty subset of a Banach algebra A. Then

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The canonical homomorphism  $\pi:A\to A/I$  is defined by  $\pi(a)=a+I$  for  $a\in A$ .

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If  $\mathcal{R}(I)$  denotes the collection of Riesz elements relative to I, then  $\mathcal{R}(I) = \pi^{-1}(\mathrm{QN}(A/I))$ .



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Let *I* be an ideal in a Banach algebra *A*. A function  $\tau: I \to \mathbb{C}$  is called a *trace* on *I* if

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If *I* is a trace ideal in *A*, then define an index function  $\iota$  on  $\Phi(I)$ .

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Suppose that  $X \subseteq Y \subseteq A$  and suppose that X, Y have the Jacobson Property. Then  $Y \setminus X$  has the Jacobson Property.

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### Remark

If we let Y = A in the above lemma we obtain the following important special case:

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### Lemma

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### **Theorem**

Let I be a proper subprime ideal in a Banach algebra A. Then  $A \setminus I$  is a (P1) regularity that has the Jacobson Property.

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The question is easy to answer as  $A^{-1}$  is a (P1) regularity but its complement is not an ideal in A.

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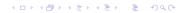
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### **Theorem**

Let A be a Banach algebra and let I be a closed ideal in A. Suppose  $\pi: A \to A/I$  is the canonical homomorphism and  $X \subseteq A/I$ .

- (i) If X has the Jacobson Property, then  $\pi^{-1}(X)$  in A has the Jacobson Property.
- (ii) If X does not have the Jacobson Property, then  $\pi^{-1}(X)$  does not have the Jacobson Property.



#### **Theorem**

Let A be a Banach algebra and let  $X \subseteq A$  be a semigroup that contains a neignbourhood of the identity. If X has the Jacobson Property, then  $\overline{X}$  and  $\partial X$  have the Jacobson Property.

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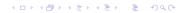
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## Corollary

Let A be a Banach algebra and let I be a closed ideal in A. Then  $\overline{X}$  and  $\partial X$  have the Jacobson Property if

$$X \in \{A_{\ell}^{-1}, A_{r}^{-1}, A_{\ell}^{-1} \cup A_{r}^{-1}, \Phi_{\ell}(I), \Phi_{r}(I), \Phi(I), \Phi_{\ell}(I) \cup \Phi_{r}(I)\}$$



## Corollary

Let A be a Banach algebra. The sets  $\overline{A^{-1}}$ ,  $\partial A^{-1}$ ,  $A \setminus \partial A^{-1}$  and  $A \setminus \overline{A^{-1}}$  all possess the Jacobson Property.

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The set  $A_{ai}$  of almost invertible elements in A is a regularity.

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The set  $\{a \in A : \sigma(a) = \{1\}\}$  has the Jacobson Property.



Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with Soc  $A \subseteq I \subseteq \operatorname{kh}(I)$ . Now one defines an abstract index function  $\iota$  was defined on the set  $\Phi(I)$  of Fredholm elements relative to I.

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$$\iota(a) = \begin{cases} -\infty & \text{if } a \in \Phi_{\ell}(I) \setminus \Phi(I) \\ \infty & \text{if } a \in \Phi_{r}(I) \setminus \Phi(I) \end{cases}$$

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If  $Z \subseteq \mathbb{Z}$ , let

$$\Phi_{Z}(I) = \{ a \in \Phi_{\ell}(I) \cup \Phi_{r}(I) : \iota(a) \in Z \}.$$

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By the above Corollary, if  $Z = \{k\}$  with  $k \in \mathbb{Z}$ , then  $\Phi_k(I)$  has the Jacobson Property. In particular, if k = 0, then  $\Phi_0(I)$  is an upper semiregularity with the Jacobson property. Also,  $\Phi_0 = \mathcal{W}(I)$  with  $\mathcal{W}(I)$  the collection of Weyl elements in A relative to I.

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For  $a, b \in A$ , we call ab - ba the *commutator* of a and b, denoted by [a, b]. Let  $A_D$  be the smallest ideal that contains all the commutators - the *commutator ideal* of A.

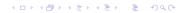
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#### **Theorem**

If A is a Banach algebra then  $A_D$  and  $A \setminus A_D$  have the Jacobson Property.



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#### **Theorem**

Let A be a complex unital Banach algebra and let f be a multiplicative linear functional on A. Then the set  $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$  is a partition of A and every member of P has the Jacobson Property.

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## Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then  $\ker(f)$  has the Jacobson Property.

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Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and  $B \subseteq \mathbb{C}$ ,  $B \neq \emptyset$ . Then  $f^{-1}(B)$  has the Jacobson Property.

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Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then ker(f) is a proper, non-trivial subprime ideal of A.

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## Proposition

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Let  $A = \mathcal{L}(\ell^2)$  and let  $R = A \setminus QN(A)$ . Then R is a lower semiregularity that does not have the Jacobson Property.

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Let A be a Banach algebra. The proof of the above example can be adapted to show that  $A \setminus Rad A$  does not have the Jacobson Property.



### **Theorem**

Let A be a noncommutative Banach algebra with  $A_{\ell}^{-1} \setminus A^{-1} \neq \emptyset$  or  $A_r^{-1} \setminus A^{-1} \neq \emptyset$ . Then Rad A does not have the Jacobson Property.

# Corollary

Let A be a noncommutative Banach algebra and let I be a closed ideal in A. Then the ideal kh(I) does not have the Jacobson Property.