The Jacobson Property in Banach algebras

A Swartz, H Raubenheimer

In a Banach algebra A

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$$

for elements $a, b \in A$.

In a Banach algebra A

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$$

for elements $a, b \in A$.

In A this looks like:

$$1 - ab \in A^{-1} \implies 1 - ba \in A^{-1}$$

In a Banach algebra A

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$$

for elements $a, b \in A$.

In A this looks like:

$$1 - ab \in A^{-1} \implies 1 - ba \in A^{-1}$$

In this talk we discuss subsets of A that have the Jacobson Property, i.e. $X \subseteq A$ such that for $a, b \in A$:

$$1 - ab \in X \implies 1 - ba \in X.$$

Preliminaries

A will be a Banach algebra, complex and unital.

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

acc K - the set of accumulation points of K,

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

acc K - the set of accumulation points of K,

iso K - the set of isolated points of K,

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

acc K - the set of accumulation points of K,

iso K - the set of isolated points of K,

∂K - topological boundary of K,

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

acc K - the set of accumulation points of K,

iso K - the set of isolated points of K,

∂K - topological boundary of K,

$\overline{K}$ - the closure of K.

Preliminaries

A will be a Banach algebra, complex and unital.

A^{-1} - the group of invertible elements in A.

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$$

the ordinary spectrum of $a \in A$ - write simply $\sigma(a)$.

A_ℓ^{-1} and A_r^{-1} - the left and right invertibles in A.

If $K \subseteq \mathbb{C}$, then

acc K - the set of accumulation points of K,

iso K - the set of isolated points of K,

∂K - topological boundary of K,

$\overline{K}$ - the closure of K.

If $X \subseteq A$ we say that X has the Jacobson Property, if for $a, b \in A$:

$$1 - ab \in X \implies 1 - ba \in X. \tag{1}$$

By an ideal in A we mean a two-sided ideal.

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

By an ideal in *A* we mean a two-sided ideal.

An ideal *J* in *A* is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

$\mathcal{I}(A)$ - the collection of all inessential elements in *A*.

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

$\mathcal{I}(A)$ - the collection of all inessential elements in A.

An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$.

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

$\mathcal{I}(A)$ - the collection of all inessential elements in A.

An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$.

$\mathrm{QN}(A)$ - the collection of quasinilpotents in A.

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

$\mathcal{I}(A)$ - the collection of all inessential elements in A.

An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$.

$\mathrm{QN}(A)$ - the collection of quasinilpotents in A.

We say $a \in A$ is *almost invertible* if $0 \notin \operatorname{acc} \sigma(a)$.

By an ideal in A we mean a two-sided ideal.

An ideal J in A is said to be *inessential*

$$a \in J \implies \operatorname{acc} \sigma(a) \subseteq \{0\},$$

$a \in J \implies \sigma(a)$ is either finite or a sequence converging to 0.

$\mathcal{I}(A)$ - the collection of all inessential elements in A.

An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$.

$\mathrm{QN}(A)$ - the collection of quasinilpotents in A.

We say $a \in A$ is *almost invertible* if $0 \notin \operatorname{acc} \sigma(a)$.

A_{ai} - the set of almost invertible elements in A.

$a \in A$ is *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$,

$$\widehat{A} = \{a \in A \,:\, a \in aAa\} \tag{2}$$

$a \in A$ is *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$,

$$\widehat{A} = \{a \in A \,:\, a \in aAa\} \tag{2}$$

These include both the left and right invertible elements,

$$A_\ell^{-1} \cup A_r^{-1} \subseteq \widehat{A} \tag{3}$$

$a \in A$ is *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$,

$$\widehat{A} = \{a \in A \,:\, a \in aAa\} \tag{2}$$

These include both the left and right invertible elements,

$$A_\ell^{-1} \cup A_r^{-1} \subseteq \widehat{A} \tag{3}$$

as well as the idempotents,

$$A^\bullet = \{a \in A \,:\, a = a^2\}. \tag{4}$$

$a \in A$ is *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$,

$$\widehat{A} = \{a \in A : a \in aAa\} \tag{2}$$

These include both the left and right invertible elements,

$$A_\ell^{-1} \cup A_r^{-1} \subseteq \widehat{A} \tag{3}$$

as well as the idempotents,

$$A^\bullet = \{a \in A : a = a^2\}. \tag{4}$$

The *decomposably regular* elements are those which admit invertible generalised inverses

$a \in A$ is *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$,

$$\widehat{A} = \{a \in A : a \in aAa\} \tag{2}$$

These include both the left and right invertible elements,

$$A_\ell^{-1} \cup A_r^{-1} \subseteq \widehat{A} \tag{3}$$

as well as the idempotents,

$$A^\bullet = \{a \in A : a = a^2\}. \tag{4}$$

The *decomposably regular* elements are those which admit invertible generalised inverses

they are those elements which can be written as a product of an invertible and an idempotent:

$$A^{-1}A^\bullet = A^\bullet A^{-1} = \{a \in A : a \in aA^{-1}a\} \subseteq \widehat{A}. \tag{5}$$

Notice that

$$A_\ell^{-1} \cap A_r^{-1} = A_\ell^{-1} \cap A^{-1}A^\bullet = A_r^{-1} \cap A^{-1}A^\bullet = A^{-1}. \qquad (6)$$

Notice that

$$A_\ell^{-1} \cap A_r^{-1} = A_\ell^{-1} \cap A^{-1}A^\bullet = A_r^{-1} \cap A^{-1}A^\bullet = A^{-1}. \tag{6}$$

If A is a Banach algebra, then

$$A^{-1}A^\bullet = \widehat{A} \cap \overline{A^{-1}}. \tag{7}$$

Notice that

$$A_\ell^{-1} \cap A_r^{-1} = A_\ell^{-1} \cap A^{-1}A^\bullet = A_r^{-1} \cap A^{-1}A^\bullet = A^{-1}. \tag{6}$$

If A is a Banach algebra, then

$$A^{-1}A^\bullet = \widehat{A} \cap \overline{A^{-1}}. \tag{7}$$

Definition

Let I be an ideal in a Banach algebra A. I is *subprime* if $ab \in I \implies a \in I$ or $b \in I$.

Notice that

$$A_\ell^{-1} \cap A_r^{-1} = A_\ell^{-1} \cap A^{-1}A^\bullet = A_r^{-1} \cap A^{-1}A^\bullet = A^{-1}. \qquad (6)$$

If A is a Banach algebra, then

$$A^{-1}A^\bullet = \widehat{A} \cap \overline{A^{-1}}. \qquad (7)$$

Definition

Let I be an ideal in a Banach algebra A. I is *subprime* if $ab \in I \implies a \in I$ or $b \in I$.

The notion of a subprime ideal defined above coincides with that of a *prime* ideal in a commutative ring or algebra.

Also called *completely prime*.

From V. Müller - 1990's :

Definition

A nonempty subset R of a Banach algebra A is called a *regularity* if it satisfies the following two conditions:

From V. Müller - 1990's :

Definition

A nonempty subset R of a Banach algebra A is called a *regularity* if it satisfies the following two conditions:

(i) if $a \in A, n \in \mathbb{N}$ then $a \in R \iff a^n \in R$;

(ii) if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ then $a, b \in R \iff ab \in R$.

It is easy to see that the following theorem holds.

It is easy to see that the following theorem holds.

Theorem

Let R be a nonempty subset of a Banach algebra A satisfying:

$$ab \in R \iff a, b \in R \tag{P1}$$

for all commuting elements $a, b \in A$. Then R is a regularity.

It is easy to see that the following theorem holds.

Theorem

Let R be a nonempty subset of a Banach algebra A satisfying:

$$ab \in R \iff a, b \in R \tag{P1}$$

for all commuting elements $a, b \in A$. Then R is a regularity.

The definition of a regularity can be divided into two parts as follows:

Definition

A nonempty subset R in a Banach algebra A will be called a *lower semiregularity* if it satisfies the following conditions:

It is easy to see that the following theorem holds.

Theorem

Let R be a nonempty subset of a Banach algebra A satisfying:

$$ab \in R \iff a, b \in R \qquad (P1)$$

for all commuting elements $a, b \in A$. Then R is a regularity.

The definition of a regularity can be divided into two parts as follows:

Definition

A nonempty subset R in a Banach algebra A will be called a *lower semiregularity* if it satisfies the following conditions:

(i) $a \in A$, $n \in \mathbb{N}$ and $a^n \in R \Rightarrow a \in R$,

(ii) if a, b, c, d are commuting elements of A and $ac + bd = 1$, then $ab \in R \Rightarrow a \in R$ and $b \in R$.

If a nonempty subset R in A satisfies

If a nonempty subset R in A satisfies

$$a, b \in A, ab = ba, ab \in R \Rightarrow a \in R \text{ and } b \in R \qquad (P1')$$

then it is a lower semiregularity

If a nonempty subset R in A satisfies

$$a, b \in A, ab = ba, ab \in R \;\Rightarrow\; a \in R \text{ and } b \in R \qquad (P1')$$

then it is a lower semiregularity

Definition

A nonempty subset R in a Banach algebra A is called an *upper semiregularity* if it satisfies the following conditions:

If a nonempty subset R in A satisfies

$$a, b \in A, ab = ba, ab \in R \Rightarrow a \in R \text{ and } b \in R \qquad (P1')$$

then it is a lower semiregularity

Definition

A nonempty subset R in a Banach algebra A is called an *upper semiregularity* if it satisfies the following conditions:

(i) $a \in R,\ n \in \mathbb{N} \Rightarrow a^n \in R$,

(ii) if a, b, c, d are commuting elements of A and $ac + bd = 1$, then $a, b \in R \Rightarrow ab \in R$,

(iii) R contains a neighbourhood of the unit element 1.

If a nonempty subset R in A satisfies

$$a, b \in A, ab = ba, ab \in R \;\Rightarrow\; a \in R \text{ and } b \in R \qquad (P1')$$

then it is a lower semiregularity

Definition

A nonempty subset R in a Banach algebra A is called an *upper semiregularity* if it satisfies the following conditions:

(i) $a \in R,\; n \in \mathbb{N} \;\Rightarrow\; a^n \in R$,

(ii) if a, b, c, d are commuting elements of A and $ac + bd = 1$, then $a, b \in R \;\Rightarrow\; ab \in R$,

(iii) R contains a neighbourhood of the unit element 1.

any semigroup containing a neighbourhood of the identity is an upper semiregularity,

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let R be a nonempty subset of a Banach algebra A. Then

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let R be a nonempty subset of a Banach algebra A. Then

$$R \text{ has the J Property} \quad \Longleftrightarrow \quad \sigma_R(ab)\setminus\{0\} = \sigma_R(ba)\setminus\{0\}$$

for all $a, b \in A$

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let R be a nonempty subset of a Banach algebra A. Then

$$R \text{ has the J Property} \iff \sigma_R(ab)\backslash\{0\} = \sigma_R(ba)\backslash\{0\}$$

for all $a, b \in A$

There are many examples of subsets in a Banach algebra possessing the Jacobson Property.

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let R be a nonempty subset of a Banach algebra A. Then

$$R \text{ has the J Property} \quad \Longleftrightarrow \quad \sigma_R(ab)\backslash\{0\} = \sigma_R(ba)\backslash\{0\}$$

for all $a, b \in A$

There are many examples of subsets in a Banach algebra possessing the Jacobson Property.

Let A be a Banach algebra and I a closed ideal in A.

Let $R \subseteq A$ - assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

called the *spectrum corresponding* to R.

If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties.

Let R be a nonempty subset of a Banach algebra A. Then

$$R \text{ has the J Property} \quad \Longleftrightarrow \quad \sigma_R(ab)\backslash\{0\} = \sigma_R(ba)\backslash\{0\}$$

for all $a, b \in A$

There are many examples of subsets in a Banach algebra possessing the Jacobson Property.

Let A be a Banach algebra and I a closed ideal in A.

The canonical homomorphism $\pi : A \to A/I$ is defined by $\pi(a) = a + I$ for $a \in A$.

We define the sets:

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

and

$$\Phi(I) = \{a \in A : a + I \in (A/I)^{-1}\}.$$

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

and

$$\Phi(I) = \{a \in A : a + I \in (A/I)^{-1}\}.$$

Denote the ideal $\pi^{-1}(\mathrm{Rad}(A/I))$ by $\mathrm{kh}(I)$.

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

and

$$\Phi(I) = \{a \in A : a + I \in (A/I)^{-1}\}.$$

Denote the ideal $\pi^{-1}(\mathrm{Rad}(A/I))$ by $\mathrm{kh}(I)$.

This is the largest ideal consisting of Riesz elements relative to the ideal I.

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

and

$$\Phi(I) = \{a \in A : a + I \in (A/I)^{-1}\}.$$

Denote the ideal $\pi^{-1}(\mathrm{Rad}(A/I))$ by $\mathrm{kh}(I)$.

This is the largest ideal consisting of Riesz elements relative to the ideal I.

We say that $a \in A$ is *Riesz relative to I* if $a + I \in \mathrm{QN}(A/I)$.

We define the sets:

$$\Phi_\ell(I) = \{a \in A : a + I \in (A/I)_\ell^{-1}\},$$

$$\Phi_r(I) = \{a \in A : a + I \in (A/I)_r^{-1}\},$$

and

$$\Phi(I) = \{a \in A : a + I \in (A/I)^{-1}\}.$$

Denote the ideal $\pi^{-1}(\mathrm{Rad}(A/I))$ by $\mathrm{kh}(I)$.

This is the largest ideal consisting of Riesz elements relative to the ideal I.

We say that $a \in A$ is *Riesz relative to I* if $a + I \in \mathrm{QN}(A/I)$.

If $\mathcal{R}(I)$ denotes the collection of Riesz elements relative to I, then $\mathcal{R}(I) = \pi^{-1}(\mathrm{QN}(A/I))$.

It is easy to see that

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if

(i) τ is linear,

(ii) $\tau(p) = 1$ for every rank one idempotent $p \in I$ and

(iii) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in A$.

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if

(i) τ is linear,
(ii) $\tau(p) = 1$ for every rank one idempotent $p \in I$ and
(iii) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in A$.

An ideal I with trace is a *trace ideal*.

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if

(i) τ is linear,

(ii) $\tau(p) = 1$ for every rank one idempotent $p \in I$ and

(iii) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in A$.

An ideal I with trace is a *trace ideal*.

If I is a trace ideal in A, then define an index function ι on $\Phi(I)$.

It is easy to see that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove that

$$I \subseteq \mathrm{kh}(I) \subseteq \mathcal{R}(I) \subseteq \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if

(i) τ is linear,

(ii) $\tau(p) = 1$ for every rank one idempotent $p \in I$ and

(iii) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in A$.

An ideal I with trace is a *trace ideal*.

If I is a trace ideal in A, then define an index function ι on $\Phi(I)$.

The socle of A is $\mathrm{Soc}(A)$.

Algebraic and Topological results

Algebraic and Topological results

In this section we exhibit basic properties of sets having the Jacobson Property.

Lemma

Suppose that $X \subseteq Y \subseteq A$ *and suppose that* X, Y *have the Jacobson Property. Then* $Y \setminus X$ *has the Jacobson Property.*

Algebraic and Topological results

In this section we exhibit basic properties of sets having the Jacobson Property.

Lemma

Suppose that $X \subseteq Y \subseteq A$ *and suppose that* X, Y *have the Jacobson Property. Then* $Y \setminus X$ *has the Jacobson Property.*

Remark

If we let $Y = A$ in the above lemma we obtain the following important special case:

If $X \subseteq A$ and X has the Jacobson Property, then $A \setminus X$ has the Jacobson Property.

Algebraic and Topological results

In this section we exhibit basic properties of sets having the Jacobson Property.

Lemma

Suppose that $X \subseteq Y \subseteq A$ and suppose that X, Y have the Jacobson Property. Then $Y \setminus X$ has the Jacobson Property.

Remark

If we let $Y = A$ in the above lemma we obtain the following important special case:

If $X \subseteq A$ and X has the Jacobson Property, then $A \setminus X$ has the Jacobson Property.

Lemma

Every proper subprime ideal has the Jacobson property.

Theorem

Let I be a proper subprime ideal in a Banach algebra A. Then $A \setminus I$ is a $(P1)$ regularity that has the Jacobson Property.

Algebraic and Topological results

In this section we exhibit basic properties of sets having the Jacobson Property.

Lemma

Suppose that $X \subseteq Y \subseteq A$ and suppose that X, Y have the Jacobson Property. Then $Y \setminus X$ has the Jacobson Property.

Remark

If we let $Y = A$ in the above lemma we obtain the following important special case:

If $X \subseteq A$ and X has the Jacobson Property, then $A \setminus X$ has the Jacobson Property.

Lemma

Every proper subprime ideal has the Jacobson property.

Theorem

Let I be a proper subprime ideal in a Banach algebra A. Then $A \setminus I$ is a $(P1)$ regularity that has the Jacobson Property.

Later we construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Later we construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Question - is every $(P1)$ regularity the complement of a proper subprime ideal?

Later we construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Question - is every $(P1)$ regularity the complement of a proper subprime ideal?

The question is easy to answer as A^{-1} is a $(P1)$ regularity but its complement is not an ideal in A.

Later we construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Question - is every $(P1)$ regularity the complement of a proper subprime ideal?

The question is easy to answer as A^{-1} is a $(P1)$ regularity but its complement is not an ideal in A.

Theorem

Let A be a Banach algebra and let I be a closed ideal in A. Suppose $\pi : A \to A/I$ is the canonical homomorphism and $X \subseteq A/I$.

Later we construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Question - is every $(P1)$ regularity the complement of a proper subprime ideal?

The question is easy to answer as A^{-1} is a $(P1)$ regularity but its complement is not an ideal in A.

Theorem

Let A be a Banach algebra and let I be a closed ideal in A. Suppose $\pi : A \to A/I$ is the canonical homomorphism and $X \subseteq A/I$.

(i) *If X has the Jacobson Property, then $\pi^{-1}(X)$ in A has the Jacobson Property.*

(ii) *If X does not have the Jacobson Property, then $\pi^{-1}(X)$ does not have the Jacobson Property.*

Theorem

Let A be a Banach algebra and let $X \subseteq A$ be a semigroup that contains a neignbourhood of the identity. If X has the Jacobson Property, then $\overline{X}$ and ∂X have the Jacobson Property.

Theorem

Let A be a Banach algebra and let $X \subseteq A$ be a semigroup that contains a neignbourhood of the identity. If X has the Jacobson Property, then $\overline{X}$ and ∂X have the Jacobson Property.

Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

The sets A_ℓ^{-1}, A_r^{-1} and A^{-1} have the Jacobson Property.

Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

The sets A_ℓ^{-1}, A_r^{-1} and A^{-1} have the Jacobson Property.

If I is a closed ideal in A, then in the quotient Banach algebra A/I, the sets $(A/I)_\ell^{-1}$, $(A/I)_r^{-1}$ and $(A/I)^{-1}$ have the Jacobson Property.

Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

The sets A_ℓ^{-1}, A_r^{-1} and A^{-1} have the Jacobson Property.

If I is a closed ideal in A, then in the quotient Banach algebra A/I, the sets $(A/I)_\ell^{-1}$, $(A/I)_r^{-1}$ and $(A/I)^{-1}$ have the Jacobson Property.

Corollary

Let A be a Banach algebra and let I be a closed ideal in A. Then $\overline{X}$ and ∂X have the Jacobson Property if

Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

The sets A_ℓ^{-1}, A_r^{-1} and A^{-1} have the Jacobson Property.

If I is a closed ideal in A, then in the quotient Banach algebra A/I, the sets $(A/I)_\ell^{-1}$, $(A/I)_r^{-1}$ and $(A/I)^{-1}$ have the Jacobson Property.

Corollary

Let A be a Banach algebra and let I be a closed ideal in A. Then $\overline{X}$ and ∂X have the Jacobson Property if

$$X \in \{A_\ell^{-1}, A_r^{-1}, A_\ell^{-1} \cup A_r^{-1}, \Phi_\ell(I), \Phi_r(I), \Phi(I), \Phi_\ell(I) \cup \Phi_r(I)\}$$

The *boundary spectrum* is generated by the set $R = A \setminus \partial A^{-1}$ - has the Jacobson Property.

Corollary

Let A be a Banach algebra. The sets $\overline{A^{-1}}, \partial A^{-1}, A \setminus \partial A^{-1}$ and $A \setminus \overline{A^{-1}}$ all possess the Jacobson Property.

The *boundary spectrum* is generated by the set $R = A \setminus \partial A^{-1}$ - has the Jacobson Property.

Corollary

Let A be a Banach algebra. The sets $\overline{A^{-1}}, \partial A^{-1}, A \setminus \partial A^{-1}$ and $A \setminus \overline{A^{-1}}$ all possess the Jacobson Property.

The set A_{ai} of almost invertible elements in A is a regularity.

Theorem

The regularity A_{ai} has the Jacobson Property.

The *boundary spectrum* is generated by the set $R = A \setminus \partial A^{-1}$ - has the Jacobson Property.

Corollary

Let A be a Banach algebra. The sets $\overline{A^{-1}}, \partial A^{-1}, A \setminus \partial A^{-1}$ and $A \setminus \overline{A^{-1}}$ all possess the Jacobson Property.

The set A_{ai} of almost invertible elements in A is a regularity.

Theorem

The regularity A_{ai} has the Jacobson Property.

Theorem

The set $\mathcal{I}(A)$ of inessential elements in a Banach algebra A has the Jacobson Property.

The *boundary spectrum* is generated by the set $R = A \setminus \partial A^{-1}$ - has the Jacobson Property.

Corollary

Let A be a Banach algebra. The sets $\overline{A^{-1}}, \partial A^{-1}, A \setminus \partial A^{-1}$ and $A \setminus \overline{A^{-1}}$ all possess the Jacobson Property.

The set A_{ai} of almost invertible elements in A is a regularity.

Theorem

The regularity A_{ai} has the Jacobson Property.

Theorem

The set $\mathcal{I}(A)$ of inessential elements in a Banach algebra A has the Jacobson Property.

Theorem

The set $\{a \in A : \sigma(a) = \{1\}\}$ has the Jacobson Property.

It is known that the set $\widehat{A}$ has the Jacobson Property. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property.

It is known that the set $\widehat{A}$ has the Jacobson Property. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property.

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. Now one defines an abstract index function ι was defined on the set $\Phi(I)$ of Fredholm elements relative to I.

It is known that the set $\widehat{A}$ has the Jacobson Property. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property.

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. Now one defines an abstract index function ι was defined on the set $\Phi(I)$ of Fredholm elements relative to I.

It was shown that this abstract index function has all the desirable properties of the classical index for Fredholm operators defined on a Banach space. One can extend the domain of the abstract index function ι to the set

It is known that the set $\widehat{A}$ has the Jacobson Property. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property.

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\,A \subseteq I \subseteq \mathrm{kh}(I)$. Now one defines an abstract index function ι was defined on the set $\Phi(I)$ of Fredholm elements relative to I.

It was shown that this abstract index function has all the desirable properties of the classical index for Fredholm operators defined on a Banach space. One can extend the domain of the abstract index function ι to the set $\Phi_\ell(I) \cup \Phi_r(I)$ by defining

$$\iota(a) = \begin{cases} -\infty & \text{if } a \in \Phi_\ell(I) \setminus \Phi(I) \\ \infty & \text{if } a \in \Phi_r(I) \setminus \Phi(I) \end{cases}$$

It is known that the set $\widehat{A}$ has the Jacobson Property. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property.

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. Now one defines an abstract index function ι was defined on the set $\Phi(I)$ of Fredholm elements relative to I.

It was shown that this abstract index function has all the desirable properties of the classical index for Fredholm operators defined on a Banach space. One can extend the domain of the abstract index function ι to the set $\Phi_{\ell}(I) \cup \Phi_r(I)$ by defining

$$\iota(a) = \begin{cases} -\infty & \text{if } a \in \Phi_{\ell}(I) \setminus \Phi(I) \\ \infty & \text{if } a \in \Phi_r(I) \setminus \Phi(I) \end{cases}$$

If $Z \subseteq \mathbb{Z}$, let

$$\Phi_Z(I) = \{a \in \Phi_{\ell}(I) \cup \Phi_r(I) \; : \; \iota(a) \in Z\}.$$

Corollary

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. If $Z \subseteq \mathbb{Z}$ and $\Phi_Z(I)$ is nonempty, then $\Phi_Z(I)$ has the Jacobson Property.

Corollary

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. If $Z \subseteq \mathbb{Z}$ and $\Phi_Z(I)$ is nonempty, then $\Phi_Z(I)$ has the Jacobson Property.

By the above Corollary, if $Z = \{k\}$ with $k \in \mathbb{Z}$, then $\Phi_k(I)$ has the Jacobson Property. In particular, if $k = 0$, then $\Phi_0(I)$ is an upper semiregularity with the Jacobson property. Also, $\Phi_0 = \mathcal{W}(I)$ with $\mathcal{W}(I)$ the collection of Weyl elements in A relative to I.

Corollary

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. If $Z \subseteq \mathbb{Z}$ and $\Phi_Z(I)$ is nonempty, then $\Phi_Z(I)$ has the Jacobson Property.

By the above Corollary, if $Z = \{k\}$ with $k \in \mathbb{Z}$, then $\Phi_k(I)$ has the Jacobson Property. In particular, if $k = 0$, then $\Phi_0(I)$ is an upper semiregularity with the Jacobson property. Also, $\Phi_0 = \mathcal{W}(I)$ with $\mathcal{W}(I)$ the collection of Weyl elements in A relative to I.

For $a, b \in A$, we call $ab - ba$ the *commutator* of a and b, denoted by $[a, b]$. Let A_D be the smallest ideal that contains all the commutators - the *commutator ideal* of A.

Corollary

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\mathrm{Soc}\, A \subseteq I \subseteq \mathrm{kh}(I)$. If $Z \subseteq \mathbb{Z}$ and $\Phi_Z(I)$ is nonempty, then $\Phi_Z(I)$ has the Jacobson Property.

By the above Corollary, if $Z = \{k\}$ with $k \in \mathbb{Z}$, then $\Phi_k(I)$ has the Jacobson Property. In particular, if $k = 0$, then $\Phi_0(I)$ is an upper semiregularity with the Jacobson property. Also, $\Phi_0 = \mathcal{W}(I)$ with $\mathcal{W}(I)$ the collection of Weyl elements in A relative to I.

For $a, b \in A$, we call $ab - ba$ the *commutator* of a and b, denoted by $[a, b]$. Let A_D be the smallest ideal that contains all the commutators - the *commutator ideal* of A.

Theorem

If A is a Banach algebra then A_D and $A \setminus A_D$ have the Jacobson Property.

Multiplicative linear functionals

Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property.

Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property.

Multiplicative linear functionals - mostly used in commutative settings - there are noncommutative Banach algebras that admit nonzero multiplicative linear functionals.

Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property.

Multiplicative linear functionals - mostly used in commutative settings - there are noncommutative Banach algebras that admit nonzero multiplicative linear functionals.

Theorem

Let A be a complex unital Banach algebra and let f be a multiplicative linear functional on A. Then the set $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$ is a partition of A and every member of P has the Jacobson Property.

Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property.

Multiplicative linear functionals - mostly used in commutative settings - there are noncommutative Banach algebras that admit nonzero multiplicative linear functionals.

Theorem

Let A be a complex unital Banach algebra and let f be a multiplicative linear functional on A. Then the set $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$ is a partition of A and every member of P has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ has the Jacobson Property.

Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property.

Multiplicative linear functionals - mostly used in commutative settings - there are noncommutative Banach algebras that admit nonzero multiplicative linear functionals.

Theorem

Let A be a complex unital Banach algebra and let f be a multiplicative linear functional on A. Then the set $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$ is a partition of A and every member of P has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ has the Jacobson Property.

We can generalize the theorem above as follows:

We can generalize the theorem above as follows:

Theorem

Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and $B \subseteq \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ is a proper, non-trivial subprime ideal of A.

We can generalize the theorem above as follows:

Theorem

Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and $B \subseteq \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ is a proper, non-trivial subprime ideal of A.

We can generalize the theorem above as follows:

Theorem

Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and $B \subseteq \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ is a proper, non-trivial subprime ideal of A.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $A \setminus \ker(f)$ is a $(P1)$ regularity in A which has the Jacobson Property.

We can generalize the theorem above as follows:

Theorem

Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and $B \subseteq \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ has the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ is a proper, non-trivial subprime ideal of A.

Corollary

Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $A \setminus \ker(f)$ is a $(P1)$ regularity in A which has the Jacobson Property.

Sets that do not have the Jacobson Property

Sets that do not have the Jacobson Property

Examples of subsets in a Banach algebra that do not possess the Jacobson Property.

Sets that do not have the Jacobson Property

Examples of subsets in a Banach algebra that do not possess the Jacobson Property.

The upper semiregularity $\mathrm{Exp}A$ does not have the Jacobson Property.

Proposition

Let A be a Banach algebra. Then the upper semiregularity $\overline{\mathrm{ExpA}}$ does not have the Jacobson Property.

Sets that do not have the Jacobson Property

Examples of subsets in a Banach algebra that do not possess the Jacobson Property.

The upper semiregularity $\mathrm{Exp}A$ does not have the Jacobson Property.

Proposition

Let A be a Banach algebra. Then the upper semiregularity $\overline{\mathrm{ExpA}}$ does not have the Jacobson Property.

Example

Let $A = \mathcal{L}(\ell^2)$ and let $R = A \backslash \mathrm{QN}(A)$. Then R is a lower semiregularity that does not have the Jacobson Property.

Sets that do not have the Jacobson Property

Examples of subsets in a Banach algebra that do not possess the Jacobson Property.

The upper semiregularity $\mathrm{Exp} A$ does not have the Jacobson Property.

Proposition

Let A be a Banach algebra. Then the upper semiregularity $\overline{\mathrm{ExpA}}$ does not have the Jacobson Property.

Example

Let $A = \mathcal{L}(\ell^2)$ and let $R = A \backslash \mathrm{QN}(A)$. Then R is a lower semiregularity that does not have the Jacobson Property.

Let A be a Banach algebra. The proof of the above example can be adapted to show that $A \backslash \mathrm{Rad}\, A$ does not have the Jacobson Property.

Theorem

Let A be a noncommutative Banach algebra with $A_\ell^{-1} \setminus A^{-1} \neq \emptyset$ or $A_r^{-1} \setminus A^{-1} \neq \emptyset$. Then $\mathrm{Rad}\, A$ does not have the Jacobson Property.

Corollary

Let A be a noncommutative Banach algebra and let I be a closed ideal in A. Then the ideal $\mathrm{kh}(I)$ does not have the Jacobson Property.