A GENERALIZATION OF PETERSEN'S MATCHING THEOREM

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65^{th} Annual Congress of the
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(**joint work with Prof. Michael A. Henning**)

08 December 2022

Outline

- What is a graph?
- Vertex degrees
- Matchings in graphs
- Petersen's Theorem
- Tutte-Berge Formula
- Matchings in 2-connected graphs of even regularity
- Matchings in 2-connected graphs of odd regularity
- Matchings in 2-connected non-regular graphs

What is a graph?

Definition (A graph)

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- A **graph** **G** is a finite nonempty set of objects, called **vertices** (the singular is **vertex**), together with a (possibly empty) set of unordered pairs of distinct vertices, called **edges**.
- The set of vertices of the graph G is called the **vertex set** of G denoted by $V(G)$ and the set of edges of the graph G is called the **edge set** of G denoted by $E(G)$.

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Example

Example of a graph

- $\mathbf{V(G) = \{v_1, v_2, v_3, v_4\}}$
- $\mathbf{E(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}}$

A graph **G**

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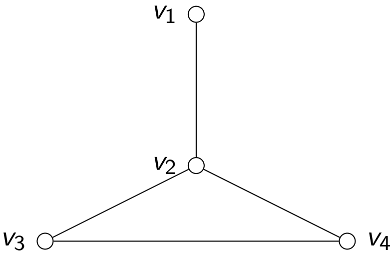

A graph **G**

Vertex degrees

Definition (Vertex Degree)

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- The **degree** of a vertex $\mathbf{v}$ is the number of edges incident with $\mathbf{v}$ denoted by $\mathbf{d_G(v)}$.
- The minimum and maximum degree of $\mathbf{G}$ is denoted by $\delta(\mathbf{G})$ and $\Delta(\mathbf{G})$, respectively.
- $\mathbf{G}$ is $\mathbf{k}$-regular if $\delta(\mathbf{G}) = \Delta(\mathbf{G}) = \mathbf{k}$.
- A $\mathbf{3}$-regular graph is also called a **cubic graph**.

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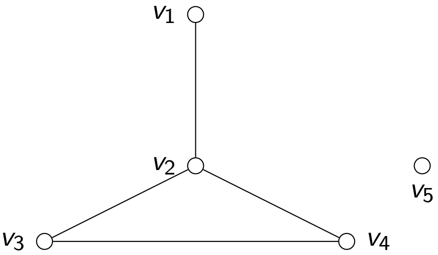

A graph **G**

The degrees of vertices of G

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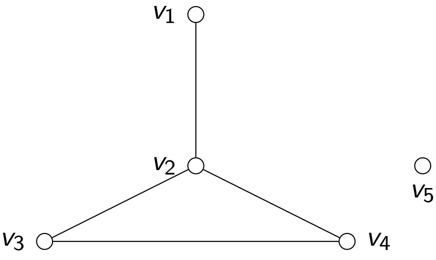

A graph **G**

The degrees of vertices of G

- $d_G(v_1) = 1$
- $d_G(v_2) = 3$
- $d_G(v_i) = 2$ for $i = 3, 4$
- $d_G(v_5) = 0$
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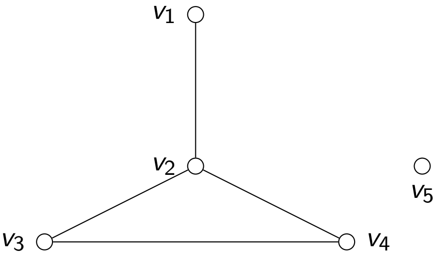

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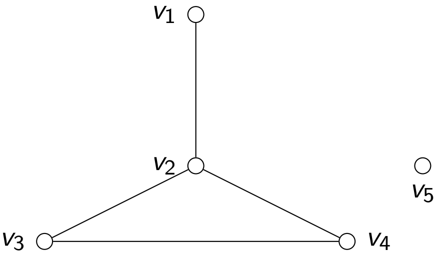

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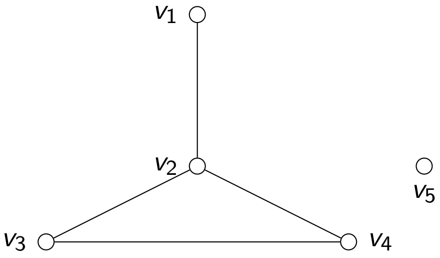

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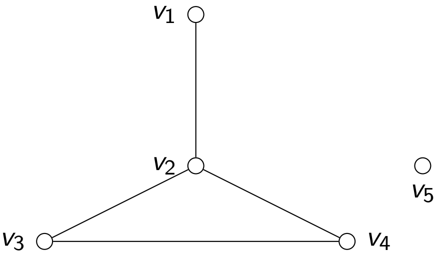

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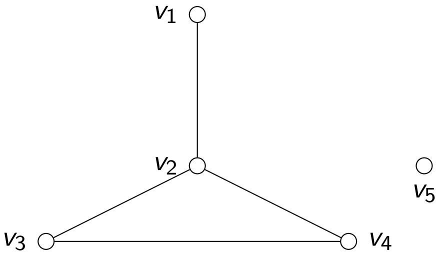

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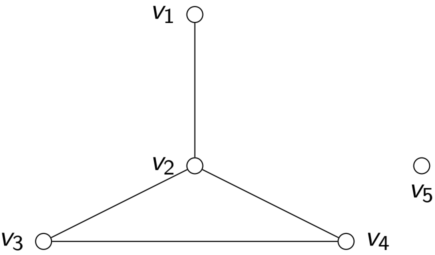

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- A set of edges in a graph **G** is **independent** if no two edges in it are adjacent in **G**; that is, an independent edge set is a set of edges without common vertices.
- A **matching** in a graph **G** is a set of independent edges.
- A **perfect matching** is a matching **M** such that every vertex of **G** is incident with an edge of **M**.
- The **matching number** of **G**, denoted by $\alpha'(G)$, is the maximum cardinality of a matching in **G**.

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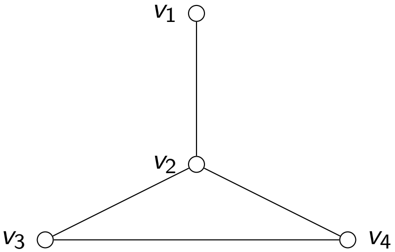

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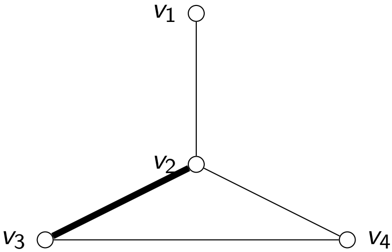
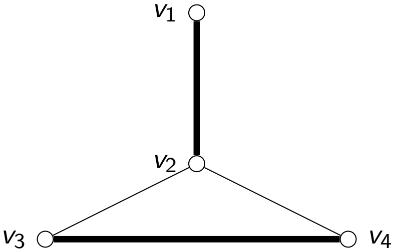


A matching of size **1**.

Example



A matching of size **2**. Hence, $\alpha'(\mathbf{G}) = \mathbf{2}$.

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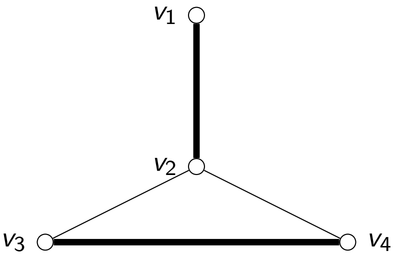

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Petersen's Theorem

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Theorem 1 (Petersen, 1891)

Every **cubic, bridgeless graph** contains a perfect matching.

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The **vertex-connectivity** of any graph is at most its **edge-connectivity**, which in turn is at most its **minimum degree**.

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Petersen's Theorem

- If a connected cubic graph has a **vertex cut** of cardinality $\mathbf{i}$, then it has an **edge cut** of cardinality $\mathbf{i}$, for $\mathbf{i} = \mathbf{1}, \mathbf{2}$.
- Hence, by the result of Whitney, we infer that the vertex-connectivity and edge-connectivity of a connected cubic graph are equal.

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If $\mathbf{G}$ is a **2-connected 3-regular graph** of order $\mathbf{n}$, then $\alpha'(\mathbf{G}) = \frac{1}{2}\mathbf{n}$.

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Tutte-Berge Formula

- Let G be a graph.
  - $G - S$ = the graph obtained from G by deleting the vertices in S and all edges incident with vertices in S for some $S \subseteq V(G)$.
  - $oc(G)$ = the number of odd components in G.

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Matchings in 2-connected graphs of even regularity

M. A. Henning, **A. Yeo**, Tight lower bounds on the size of a matching in a regular graph. **Graphs Combin. 23** (2007), 647–657.

Theorem 5 (MAH & Yeo, 2007)

For $k \geq 4$ even, if G is a connected k-regular graph of order n, then

$$\alpha'(G) \geq \min\left\{\left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{n}{2}, \frac{n-1}{2}\right\},$$

and this bound is tight.

All graphs achieving equality on this bound are 2-connected!

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A $\mathbf{k}$-diamond $\mathbf{D_k}$

- For $k \geq 3$, $D_k = K_{k+1} - e$ where e is an edge of K_{k+1}.
- The vertex of degree $k-1$ in D_k is called a **link vertex**.

Figure: The 4-diamond D_4

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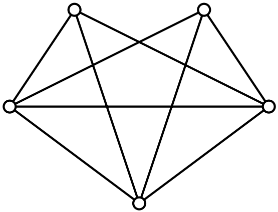

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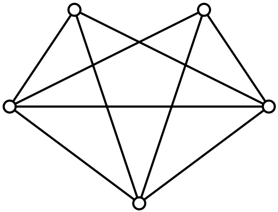

Figure: The 4-diamond D_4

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Matchings in 2-connected graphs of even regularity

A bipartite graph $\mathbf{B}_{\mathbf{k},\ell}$

For $\mathbf{k} \geq \mathbf{4}$ even and $\ell \geq \mathbf{1}$, let $\mathbf{B}_{\mathbf{k},\ell}$ be a connected bipartite graph of order $\mathbf{n} = (\mathbf{k}+\mathbf{2})\ell$ with partite sets $\mathbf{X}$ and $\mathbf{Y}$, where $|\mathbf{X}| = \mathbf{2}\ell$ and $|\mathbf{Y}| = \mathbf{k}\ell$, and where every vertex in $\mathbf{X}$ has degree $\mathbf{k}$ and every vertex in $\mathbf{Y}$ has degree $\mathbf{2}$.

Figure: The bipartite graph $\mathbf{B}_{4,1}$

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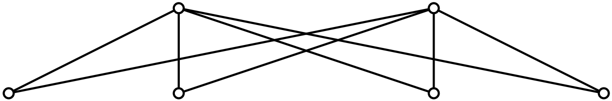

Figure: The bipartite graph $\mathbf{B_{4,1}}$

Matchings in 2-connected graphs of even regularity

The family $\mathcal{H}_{\mathbf{k},2\mathrm{conn}}$

- Let $\mathbf{H_{k,\ell}}$ be obtained from $\mathbf{B_{k,\ell}}$ as follows:

- The graph $\mathbf{B_{k,\ell}}$ is called the **underlying bipartite graph** of $\mathbf{H_{k,\ell}}$.

Figure: The graph $\mathbf{H_{4,1}}$ in the family $\mathcal{H}_{\mathbf{4},2\mathrm{conn}}$.

Matchings in 2-connected graphs of even regularity

The family $\mathcal{H}_{\mathbf{k},\mathrm{2conn}}$

- Let $\mathbf{H_{k,\ell}}$ be obtained from $\mathbf{B_{k,\ell}}$ as follows:
  - Replace every vertex $\mathbf{v} \in \mathbf{Y}$, whose two neighbors in $\mathbf{X}$ we label $\mathbf{v_1}$ and $\mathbf{v_2}$, with a copy of a $\mathbf{k}$-diamond $\mathbf{D_k}$, where we join the one link vertex of the added diamond to $\mathbf{v_1}$ and the other link vertex to $\mathbf{v_2}$.
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Figure: The graph $\mathbf{H_{4,1}}$ in the family $\mathcal{H}_{\mathbf{4},\mathrm{2conn}}$.

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The family $\mathcal{H}_{\mathbf{k},2\mathrm{conn}}$

- Let $\mathbf{H}_{\mathbf{k},\ell}$ be obtained from $\mathbf{B}_{\mathbf{k},\ell}$ as follows:
  - Replace every vertex $\mathbf{v} \in \mathbf{Y}$, whose two neighbors in $\mathbf{X}$ we label $\mathbf{v_1}$ and $\mathbf{v_2}$, with a copy of a $\mathbf{k}$-diamond $\mathbf{D_k}$, where we join the one link vertex of the added diamond to $\mathbf{v_1}$ and the other link vertex to $\mathbf{v_2}$.
- The graph $\mathbf{B}_{\mathbf{k},\ell}$ is called the **underlying bipartite graph** of $\mathbf{H}_{\mathbf{k},\ell}$.

Figure: The graph $\mathbf{H_{4,1}}$ in the family $\mathcal{H}_{\mathbf{4},2\mathrm{conn}}$.

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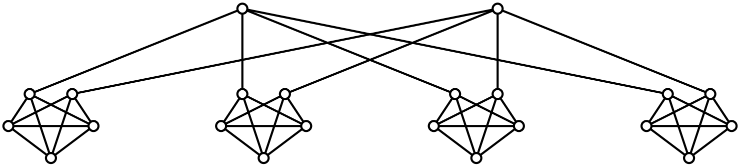

Figure: The graph $\mathbf{H_{4,1}}$ in the family $\mathcal{H}_{\mathbf{4},2\text{conn}}$.

Matchings in 2-connected graphs of even regularity

The family $\mathcal{H}_{\mathbf{k},\text{2conn}}$

- Let $\mathbf{H_{k,\ell}}$ be obtained from $\mathbf{B_{k,\ell}}$ as follows:
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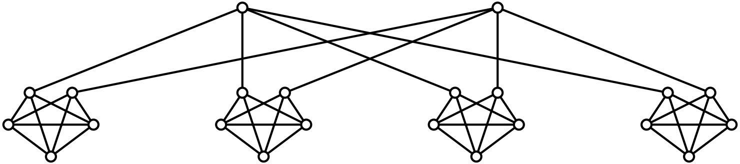

Figure: The graph $\mathbf{H_{4,1}}$ in the family $\mathcal{H}_{\mathbf{4},\text{2conn}}$.

Matchings in 2-connected graphs of even regularity

M. A. Henning, Z. B. Shozi, A characterization of graphs with given maximum degree and smallest possible matching number: **II**, **Discrete Math. 345 (3) (2022)**, 112331.

Theorem 6 (MAH & ZBS, 2022)

For $\mathbf{k} \geq \mathbf{4}$ an even integer, if $\mathbf{G}$ is a $\mathbf{2}$-connected $\mathbf{k}$-regular graph of order $\mathbf{n}$ and $\alpha'(\mathbf{G}) < \frac{1}{2}(\mathbf{n}-\mathbf{1})$, then

$$\alpha'(\mathbf{G}) \geq \left(\frac{\mathbf{k}^2+\mathbf{4}}{\mathbf{k}^2+\mathbf{k}+\mathbf{2}} \right) \times \frac{\mathbf{n}}{\mathbf{2}},$$

with equality if and only if $\mathbf{G} \in \mathcal{H}_{\mathbf{k},2\mathrm{conn}}$.

Matchings in 2-connected graphs of even regularity

Observation 1 (MAH & Yeo, 2007)

For an integer $\mathbf{k} \geq \mathbf{2}$ and a graph $\mathbf{G}$ with $\mathbf{\Delta(G)} \leq \mathbf{k}$, if

$$\sum_{\mathbf{x} \in \mathbf{V(G)}} (\mathbf{k} - \mathbf{d_G(x)}) < \mathbf{k},$$

then $|\mathbf{V(G)}| \geq \mathbf{k} + \mathbf{1}$.

Matchings in 2-connected graphs of odd regularity

- Let $\mathbf{G}$ be a graph.

  For $X, Y \subseteq V(G)$, we denote by $[X, Y]$ the set of all edges joining the set X and the set Y in G.

- For $\mathbf{r}, \mathbf{s} \geq \mathbf{1}$, a **double star** $\mathbf{S(r, s)}$ is a tree with exactly two (adjacent) vertices that are not leaves. In particular, $S(1, 2)$ is obtained from a star $K_{1,3}$ by subdividing one edge exactly once.

Matchings in 2-connected graphs of odd regularity

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Matchings in 2-connected graphs of odd regularity

A **k**-unit

- For $k \geq 5$ an odd integer,

- The vertices of degree less than k in a k-unit are called **link vertices**.
- For $i = 1, 2$, a link vertex of degree $k - i$ is called a **type-i link vertex**.

Matchings in 2-connected graphs of odd regularity

A $\mathbf{k}$-unit

- For $\mathbf{k} \geq 5$ an odd integer, if

$$\mathbf{U_1} = \mathbf{K_3} \cup \left(\frac{\mathbf{k}-\mathbf{1}}{\mathbf{2}}\right)\mathbf{K_2} \quad \text{and} \quad \mathbf{U_2} = \mathbf{S}(\mathbf{1},\mathbf{2}) \cup \left(\frac{\mathbf{k}-\mathbf{3}}{\mathbf{2}}\right)\mathbf{K_2},$$

  then

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Matchings in 2-connected graphs of odd regularity

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Matchings in 2-connected graphs of odd regularity

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Matchings in 2-connected graphs of odd regularity

The family $\mathcal{R}_k$

- For $k \geq 5$ odd and $\ell \geq 1$, let $\mathcal{R}_{k,\ell}$ be a 2-connected k-regular graph obtained from the disjoint union of $k\ell$ k-units by adding an independent set X of 3ℓ vertices and adding edges as follows.
- For $k \geq 5$ odd, let

$$\mathcal{R}_k = \bigcup_{\ell \geq 1} \mathcal{R}_{k,\ell}.$$

Matchings in 2-connected graphs of odd regularity

The family $\mathcal{R}_{\mathbf{k}}$

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    - Add an edge from each type-$\mathbf{1}$ link vertex to one vertex in $\mathbf{X}$ and add an edge from each type-$\mathbf{2}$ link vertex to two distinct vertices in $\mathbf{X}$.
    - These $\mathbf{3k}\ell$ edges are added in such a way that the resulting graph $\mathcal{R}_{\mathbf{k},\ell}$ is $\mathbf{2}$-connected and each vertex of $\mathbf{X}$ has degree $\mathbf{k}$ in $\mathcal{R}_{\mathbf{k},\ell}$.
    - By construction, the resulting graph $\mathcal{R}_{\mathbf{k},\ell}$ is a $\mathbf{2}$-connected $\mathbf{k}$-regular graph.
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Matchings in 2-connected graphs of odd regularity

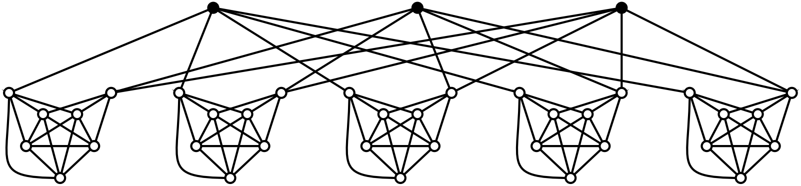
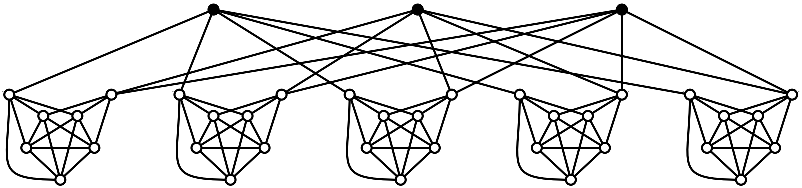


Figure: A graph **G** in the family $\mathcal{R}_{\mathbf{5},\mathbf{1}}$

- **G** is built from five **5**-units.
- Every unit has one type-**1** link vertex and one type-**2** link vertex.

Matchings in 2-connected graphs of odd regularity



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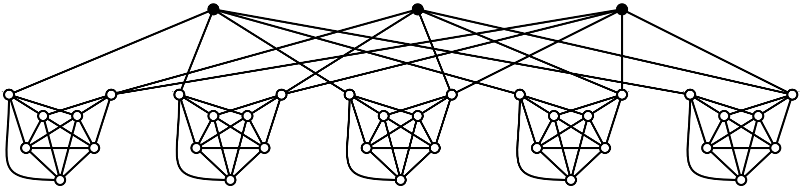
Matchings in 2-connected graphs of odd regularity



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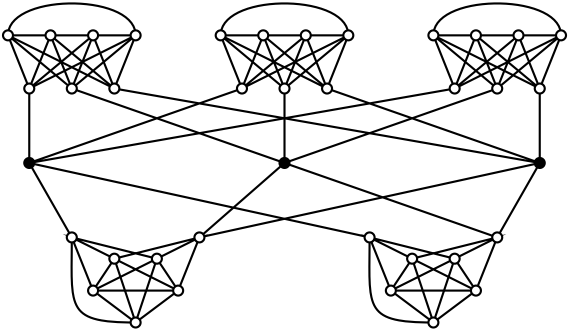

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- Two units have one type-**1** link vertex and one type-**2** link vertex.

Matchings in 2-connected graphs of odd regularity

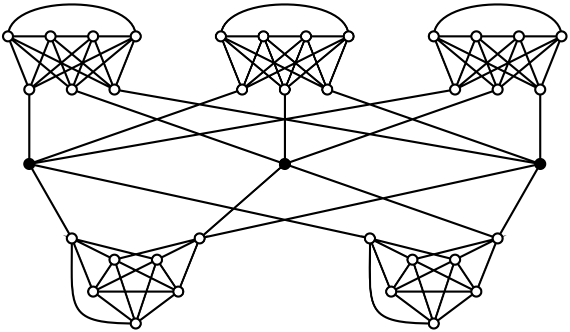

Figure: A graph $\mathbf{H}$ in the family $\mathcal{R}_{\mathbf{5},\mathbf{1}}$

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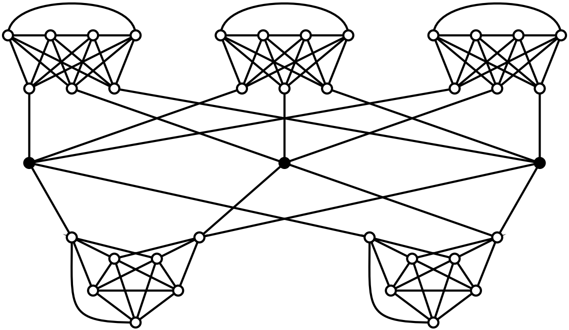
Matchings in 2-connected graphs of odd regularity



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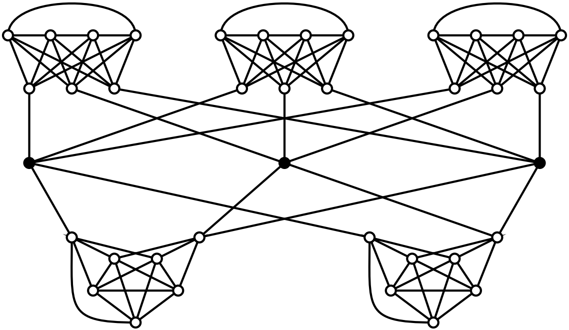
Matchings in 2-connected graphs of odd regularity



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Matchings in 2-connected graphs of odd regularity

- If $\mathbf{G} \in \mathcal{R}_{\mathbf{k}}$ for some $\mathbf{k} \geq \mathbf{5}$ odd, then $\mathbf{G} = \mathcal{R}_{\mathbf{k},\ell}$ for some $\ell \geq \mathbf{1}$.
- Moreover, if $\mathbf{G}$ has order $\mathbf{n}$, then $\mathbf{n} = \ell(\mathbf{k}^2 + 2\mathbf{k} + 3)$ and
- $\mathbf{G}$ has matching number

$$
\begin{aligned}
\alpha'(\mathbf{G}) &= |\mathbf{X}| + \frac{1}{2}(\mathbf{k}+1) \times \mathbf{k}\ell \\
&= \frac{1}{2}\ell(\mathbf{k}^2 + \mathbf{k} + 6) \\
&= \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.
\end{aligned}
$$

Observation 2

For $\mathbf{k} \geq \mathbf{5}$ an odd integer, if $\mathbf{G} \in \mathcal{R}_{\mathbf{k}}$ has order $\mathbf{n}$, then $\mathbf{G}$ is a $\mathbf{2}$-connected $\mathbf{k}$-regular graph satisfying

$$
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$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.$$

Matchings in 2-connected graphs of odd regularity

- If $\mathbf{G} \in \mathcal{R}_\mathbf{k}$ for some $\mathbf{k} \geq \mathbf{5}$ odd, then $\mathbf{G} = \mathcal{R}_{\mathbf{k},\ell}$ for some $\ell \geq \mathbf{1}$.
- Moreover, if $\mathbf{G}$ has order $\mathbf{n}$, then $\mathbf{n} = \ell(\mathbf{k}^2 + 2\mathbf{k} + 3)$ and
- $\mathbf{G}$ has matching number

$$\begin{aligned}\alpha'(\mathbf{G}) &= |\mathbf{X}| + \frac{1}{2}(\mathbf{k}+1) \times \mathbf{k}\ell \\ &= \frac{1}{2}\ell(\mathbf{k}^2 + \mathbf{k} + 6) \\ &= \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.\end{aligned}$$

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We are now in a position to present a generalization of Petersen's matching theorem.

Theorem 7 (MAH & ZBS, 2022)

For $k \geq 3$ an odd integer, if G is a 2-connected k-regular graph of order n, then

$$\alpha'(G) \geq \left(\frac{k^2+k+6}{k^2+2k+3}\right) \times \frac{n}{2}, \tag{1}$$

Proof Sketch

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- For $k \geq 3$ an odd integer, let G be a 2-connected k-regular graph of order n.
- $k = 3 \implies \alpha'(G) \geq \frac{1}{2}n \implies \alpha'(G) = \frac{1}{2}n$.

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Proof Sketch

- For $\mathbf{k} \geq \mathbf{3}$ an odd integer, let $\mathbf{G}$ be a $\mathbf{2}$-connected $\mathbf{k}$-regular graph of order $\mathbf{n}$.
- $\mathbf{k} = \mathbf{3} \implies \alpha'(\mathbf{G}) \geq \frac{1}{2}\mathbf{n} \implies \alpha'(\mathbf{G}) = \frac{1}{2}\mathbf{n}$.

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Matchings in 2-connected graphs of odd regularity

Proof Sketch

- Let $\mathbf{X} \subseteq \mathbf{V}(\mathbf{G})$ such that $\alpha'(\mathbf{G}) = \frac{1}{2}(\mathbf{n} + |\mathbf{X}| - \mathrm{oc}(\mathbf{G} - \mathbf{X}))$.
- $|\mathbf{X}| = \emptyset \implies \alpha'(\mathbf{G}) = \frac{1}{2}\mathbf{n}$, since in this case $\mathrm{oc}(\mathbf{G}) = \mathbf{0}$, noting that $\mathbf{n}$ is even. Hence, we may assume that $\mathbf{X} \neq \emptyset$.
- Since $\mathbf{G}$ is $\mathbf{2}$-connected, $\mathbf{G}$ is bridgeless, and so every odd component in $(\mathbf{G} - \mathbf{X})$ is joined in $\mathbf{G}$ to $\mathbf{X}$ by at least two edges.
- Moreover, since $\mathbf{k}$ is odd, no odd component in $(\mathbf{G} - \mathbf{X})$ is joined in $\mathbf{G}$ to $\mathbf{X}$ by exactly two edges.
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- Let
  - $\mathbf{Y} = \mathbf{V}(\mathbf{G}) \setminus \mathbf{X}$,
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  - $\mathbf{y_3}$ denote the number of odd components in $\mathbf{G} - \mathbf{X}$ that are joined in $\mathbf{G}$ to $\mathbf{X}$ by less than $\mathbf{k}$ edges.
- Thus, $\mathrm{oc}(\mathbf{G} - \mathbf{X}) = \mathbf{y_3} + \mathbf{y_k}$.
- If $[\mathbf{X}, \mathbf{Y}]$ denotes the set of all edges between $\mathbf{X}$ and $\mathbf{Y}$, then by simple counting we have

$$\mathbf{k}|\mathbf{X}| \geq |[\mathbf{X}, \mathbf{Y}]| \geq \mathbf{3y_3} + \mathbf{ky_k},$$

and so

$$|\mathbf{X}| \geq \frac{\mathbf{3y_3}}{\mathbf{k}} + \mathbf{y_k} \geq \frac{\mathbf{3y_3}}{\mathbf{k}}. \qquad (2)$$

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Matchings in 2-connected graphs of odd regularity

Proof Sketch

- If $\mathbf{H}$ is a component of $\mathbf{G}-\mathbf{X}$, then $\sum_{x\in V(H)} (k-d_H(x))$ is the number of edges between $\mathbf{H}$ and $\mathbf{X}$ in $\mathbf{G}$.
- $\sum_{x\in V(H)} (k-d_H(x)) < k \implies |V(H)| \geq k+1.$
- If $\mathbf{H}$ is an odd component of $\mathbf{G}-\mathbf{X}$, then $|V(H)| \geq k+2$.
- Hence every odd component in $\mathbf{G}-\mathbf{X}$ that is joined to $\mathbf{X}$ with less than $\mathbf{k}$ edges must contain at least $k+2$ vertices, implying that

$$n \geq |X| + y_k + y_3(k+2) \geq |X| + y_3(k+2). \tag{3}$$

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- If $\mathbf{H}$ is an odd component of $\mathbf{G} - \mathbf{X}$, then $|V(H)| \geq k + 2$.
- Hence every odd component in $G - X$ that is joined to X with less than k edges must contain at least $k + 2$ vertices, implying that

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Proof Sketch

- We therefore have the following.

$$\alpha'(\mathbf{G}) = \frac{1}{2}(\mathbf{n} + |\mathbf{X}| - \mathrm{oc}(\mathbf{G} - \mathbf{X}))$$
$$= \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3} \times \frac{\mathbf{n}}{2}\right) + \left(\frac{\mathbf{k} - 3}{\mathbf{k}^2 + 2\mathbf{k} + 3} \times \frac{\mathbf{n}}{2}\right) + \left(\frac{|\mathbf{X}| - (\mathbf{y}_3 + \mathbf{y}_\mathbf{k})}{2}\right).$$

- By Inequalities (2) and (3) we have

$$\alpha'(\mathbf{G}) \geq \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3} \times \frac{\mathbf{n}}{2}\right) + \left(\frac{\mathbf{k} - 3}{\mathbf{k}^2 + 2\mathbf{k} + 3} \times \frac{|\mathbf{X}| + \mathbf{y}_3(\mathbf{k} + 2)}{2}\right) + \left(\frac{\frac{3\mathbf{y}_3}{\mathbf{k}} + \mathbf{y}_\mathbf{k} - (\mathbf{y}_3 + \mathbf{y}_\mathbf{k})}{2}\right)$$

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- We therefore have the following.

$$\alpha'(G) = \frac{1}{2}(n + |X| - \mathrm{oc}(G - X)) = \left(\frac{k^2 + k + 6}{k^2 + 2k + 3} \times \frac{n}{2}\right) + \left(\frac{k - 3}{k^2 + 2k + 3} \times \frac{n}{2}\right) + \left(\frac{|X| - (y_3 + y_k)}{2}\right).$$

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Matchings in 2-connected graphs of odd regularity

Proof Sketch

- By Inequality (2) we have

$$\alpha'(G) \geq \left(\frac{k^2+k+6}{k^2+2k+3} \times \frac{n}{2}\right) + \left(\frac{k-3}{k^2+2k+3} \times \frac{\frac{3y_3}{k} + y_3(k+2)}{2}\right) - y_3\left(\frac{k-3}{2k}\right)$$

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$$= \left(\frac{k^2+k+6}{k^2+2k+3}\right) \times \frac{n}{2}.$$

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- Suppose that G achieves equality in Inequality (4), that is,

$$\alpha'(G) = \left(\frac{k^2+k+6}{k^2+2k+3}\right) \times \frac{n}{2}.$$

- We must have equality in both Inequalities (2) and (3).
- Equality in Inequality (2) implies that X is an independent set and every vertex in X has degree k. Furthermore, $oc(G-X) = y_3$ and every odd component in $G-X$ is joined to X by exactly three edges.
- Equality in Inequality (3) implies that every odd component in $G-X$ has order exactly $k+2$. Further, every vertex belongs to X or to an odd component of $G-X$.
- Thus, every component of $G-X$ is an odd component and is a k-unit.

Proof Sketch

- Suppose that $\mathbf{G}$ achieves equality in Inequality (4), that is,

$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.$$

- We must have equality in both Inequalities (2) and (3).
- Equality in Inequality (2) implies that $\mathbf{X}$ is an independent set and every vertex in $\mathbf{X}$ has degree $\mathbf{k}$. Furthermore, $\mathrm{oc}(\mathbf{G} - \mathbf{X}) = \mathbf{y}_3$ and every odd component in $\mathbf{G} - \mathbf{X}$ is joined to $\mathbf{X}$ by exactly three edges.
- Equality in Inequality (3) implies that every odd component in $\mathbf{G} - \mathbf{X}$ has order exactly $\mathbf{k} + 2$. Further, every vertex belongs to $\mathbf{X}$ or to an odd component of $\mathbf{G} - \mathbf{X}$.
- Thus, every component of $\mathbf{G} - \mathbf{X}$ is an odd component and is a $\mathbf{k}$-unit.

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- Suppose that $\mathbf{G}$ achieves equality in Inequality (4), that is,

$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.$$

- We must have equality in both Inequalities (2) and (3).
- Equality in Inequality (2) implies that $\mathbf{X}$ is an independent set and every vertex in $\mathbf{X}$ has degree $\mathbf{k}$. Furthermore, $\mathrm{oc}(\mathbf{G}-\mathbf{X}) = \mathbf{y}_3$ and every odd component in $\mathbf{G}-\mathbf{X}$ is joined to $\mathbf{X}$ by exactly three edges.
- Equality in Inequality (3) implies that every odd component in $\mathbf{G}-\mathbf{X}$ has order exactly $\mathbf{k}+2$. Further, every vertex belongs to $\mathbf{X}$ or to an odd component of $\mathbf{G}-\mathbf{X}$.
- Thus, every component of $\mathbf{G}-\mathbf{X}$ is an odd component and is a $\mathbf{k}$-unit.

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- Suppose that $\mathbf{G}$ achieves equality in Inequality ($\mathbf{4}$), that is,

$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k^2+k+6}}{\mathbf{k^2+2k+3}}\right) \times \frac{\mathbf{n}}{\mathbf{2}}.$$

- We must have equality in both Inequalities ($\mathbf{2}$) and ($\mathbf{3}$).
- Equality in Inequality ($\mathbf{2}$) implies that $\mathbf{X}$ is an independent set and every vertex in $\mathbf{X}$ has degree $\mathbf{k}$. Furthermore, $\mathrm{oc}(\mathbf{G}-\mathbf{X}) = \mathbf{y_3}$ and every odd component in $\mathbf{G}-\mathbf{X}$ is joined to $\mathbf{X}$ by exactly three edges.
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- Thus, every component of $\mathbf{G}-\mathbf{X}$ is an odd component and is a $\mathbf{k}$-unit.

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- Suppose that $\mathbf{G}$ achieves equality in Inequality (4), that is,

$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.$$

- We must have equality in both Inequalities (2) and (3).
- Equality in Inequality (2) implies that $\mathbf{X}$ is an independent set and every vertex in $\mathbf{X}$ has degree $\mathbf{k}$. Furthermore, $\mathrm{oc}(\mathbf{G} - \mathbf{X}) = \mathbf{y_3}$ and every odd component in $\mathbf{G} - \mathbf{X}$ is joined to $\mathbf{X}$ by exactly three edges.
- Equality in Inequality (3) implies that every odd component in $\mathbf{G} - \mathbf{X}$ has order exactly $\mathbf{k} + 2$. Further, every vertex belongs to $\mathbf{X}$ or to an odd component of $\mathbf{G} - \mathbf{X}$.
- Thus, every component of $\mathbf{G} - \mathbf{X}$ is an odd component and is a $\mathbf{k}$-unit.

Proof Sketch

- Suppose that $\mathbf{G}$ achieves equality in Inequality ($\mathbf{4}$), that is,

$$\alpha'(\mathbf{G}) = \left(\frac{\mathbf{k}^2 + \mathbf{k} + 6}{\mathbf{k}^2 + 2\mathbf{k} + 3}\right) \times \frac{\mathbf{n}}{2}.$$

- We must have equality in both Inequalities ($\mathbf{2}$) and ($\mathbf{3}$).
- Equality in Inequality ($\mathbf{2}$) implies that $\mathbf{X}$ is an independent set and every vertex in $\mathbf{X}$ has degree $\mathbf{k}$. Furthermore, $\mathrm{oc}(\mathbf{G} - \mathbf{X}) = \mathbf{y}_3$ and every odd component in $\mathbf{G} - \mathbf{X}$ is joined to $\mathbf{X}$ by exactly three edges.
- Equality in Inequality ($\mathbf{3}$) implies that every odd component in $\mathbf{G} - \mathbf{X}$ has order exactly $\mathbf{k} + 2$. Further, every vertex belongs to $\mathbf{X}$ or to an odd component of $\mathbf{G} - \mathbf{X}$.
- Thus, every component of $\mathbf{G} - \mathbf{X}$ is an odd component and is a $\mathbf{k}$-unit.

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- By the way in which the family $\mathcal{R}_{\mathbf{k}}$ is constructed, this in turn implies that $\mathbf{G} \in \mathcal{R}_{\mathbf{k}}$.
- This completes the characterization of the extremal graphs achieving equality in Inequality (4). □

Matchings in 2-connected graphs of odd regularity

Proof Sketch

- By the way in which the family $\mathcal{R}_{\mathbf{k}}$ is constructed, this in turn implies that $\mathbf{G} \in \mathcal{R}_{\mathbf{k}}$.
- This completes the characterization of the extremal graphs achieving equality in Inequality (**4**). □

Matchings in 2-connected non-regular graphs

The family $\mathcal{G}_k$

Let $\mathcal{G}_k = \{B_{k,\ell} \mid \ell \geq 1\}$.

Theorem 8 (MAH & ZBS, 2022)

For $k \geq 3$, if G is a 2-connected graph of order n and maximum degree $\Delta(G) = k$, then

$$\alpha'(G) \geq \frac{2n}{k+2}, \tag{4}$$

with equality if and only if $G \in \mathcal{G}_k$.

Matchings in 2-connected non-regular graphs

The family $\mathcal{G}_{\mathbf{k}}$

Let $\mathcal{G}_{\mathbf{k}} = \{\mathbf{B}_{\mathbf{k},\ell} \mid \ell \geq \mathbf{1}\}$.

Theorem 8 (MAH & ZBS, 2022)

For $\mathbf{k} \geq \mathbf{3}$, if $\mathbf{G}$ is a $\mathbf{2}$-connected graph of order $\mathbf{n}$ and maximum degree $\boldsymbol{\Delta}(\mathbf{G}) = \mathbf{k}$, then

$$\alpha'(\mathbf{G}) \geq \frac{\mathbf{2n}}{\mathbf{k}+\mathbf{2}}, \tag{4}$$

with equality if and only if $\mathbf{G} \in \mathcal{G}_{\mathbf{k}}$.

Matchings in 2-connected non-regular graphs

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Matchings in 2-connected non-regular graphs

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with equality if and only if $\mathbf{G} \in \mathcal{G}_{\mathbf{k}}$.

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