Representable distributive quasi relation algebras

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Outline

Some familiar operations on relations

Let X be a set and $R, S \subseteq X^2$. We will use the following binary relations:

- $\varnothing$
- $\mathsf{id}_X = \{(x,x) \mid x \in X\}$
- X^2
- $R \circ S = \{\, (a,b) \mid \exists c((a,c) \in R \,\&\, (c,b) \in S) \,\}$
- $R^{\smile} = \{\, (b,a) \mid (a,b) \in R \,\}$
- $R^{\mathsf{c}} = \{\, (a,b) \mid (a,b) \notin R \,\}$
- $R \cup S$
- $R \cap S$

Relation algebras

Definition (Chin and Tarski (1951))

A relation algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, ', \bot, \top, \cdot, 1, ^{\smile} \rangle$ such that

- $\langle A, \wedge, \vee, ', \bot, \top \rangle$ is a Boolean algebra,
- $\langle A, \cdot, 1 \rangle$ is a monoid

and, for all $a, b, c \in A$, the following hold:

1. $a^{\smile\smile} = a$
2. $(a \cdot b)^{\smile} = b^{\smile} \cdot a^{\smile}$
3. $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$
4. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
5. $a^{\smile} \cdot (a \cdot b)' \leqslant b'$

Concrete relation algebras

Theorem

Let X be a set and $E \subseteq X^2$ an equivalence relation. Then

$$\mathcal{Sb}(E) = \langle \mathcal{P}(E), \cap, \cup, {}^c, \varnothing, E, \circ, \mathsf{id}_X, {}^{\smile} \rangle$$

is a relation algebra.

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Corollary

Let X be a set. Then

$$\mathcal{R}e(X) = \langle \mathcal{P}(X^2), \cap, \cup, {}^c, \varnothing, X^2, \circ, \mathrm{id}_X, {}^{\smile} \rangle$$

is a relation algebra.

Representable relation algebras

Definition

A relation algebra $\mathbf{A}$ is representable, or in **RRA**, if it satisfies one of the following equivalent conditions:

- $\mathbf{A} \in \mathbb{IS}\{\mathcal{S}b(E) \mid E \text{ is an equivalence relation}\}$
- $\mathbf{A} \in \mathbb{ISP}\{\mathcal{R}e(X) \mid X \text{ is a set}\}$

Early studies of representability

From Bulletin of the AMS 1948:

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A *relation algebra* (RA) A is a Boolean algebra with a binary associative operator $|$ and a unary operator $^\smile$ such that: $|$ has the unit element e; $a^{\smile\smile}=a$; $(a|b)^\smile = b^\smile | a^\smile$; $a|\overline{a|b} \leqq \bar{b}$. (See Tarski, Journal of Symbolic Logic vol. 6, and Abstract 88.) Examples: I. *Proper relation algebras* (PRA's)—families A' of subrelations of a binary relation U closed under set addition $R+S$, complementation $U-R$, relative multiplication $R|S$, conversion $R^\smile$; the set of all (a, a) with $a \in U$ is in A'. II (by McKinsey). *Frobenius algebras* (FA's)—families A'' of subsets of a group G closed under set addition and complementation, complex multiplication and inversion; the smallest subgroup of G is in A''. Results: 1. A is extendible to a complete atomistic RA. 2. A is isomorphic with an algebra where $R+S$, $R|S$, $R^\smile$ (but not complementation) have the meaning in I. 3. An atomistic A where $a^\smile | a \leqq e$ (or $a^\smile | a = e$) for every atom a is isomorphic with a PRA (or FA). 4 (by McKinsey-Tarski). A is simple if and only if $a \neq 0 \rightarrow 1|a|1=1$. 5. A PRA which is simple is isomorphic with a PRA A' where U is a Cartesian set-square; and conversely. Problems: Are all RA's isomorphic with PRA's? Are all RA's with $a|b=0 \rightarrow (a=0 \vee b=0)$ isomorphic with FA's? (Received October 21, 1947.)

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- Lyndon (1950): there is a non-representable relation algebra.
- Tarski (1955): **RRA** is a variety.
- Monk (1964): **RRA** is non-finitely axiomatizable.

Quasi relation algebras (Galatos & Jipsen 2013)

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An FL-algebra is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ such that

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- $a \cdot b \leqslant c \iff b \leqslant a\backslash c \iff a \leqslant c/b$, and
- 0 an arbitrary element of A.

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Define: $\sim a = a\backslash 0 \qquad -a = 0/a \qquad a + b = \sim(-b \cdot -a)$

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Definition

An FL$'$-algebra is an FL-algebra with a unary operation $'$ such that $a'' = a$.

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Definition

An FL′-algebra is an FL-algebra with a unary operation $'$ such that $a'' = a$. A DmFL′-algebra is an FL′-algebra such that $(a \vee b)' = a' \wedge b'$. A quasi relation algebra (qRA) is a DmFL′-algebra that satisfies:

(Di) $\quad (\sim a)' = -(a')$ $\qquad$ (Dp) $\quad (a \cdot b)' = a' + b'$

More about qRAs

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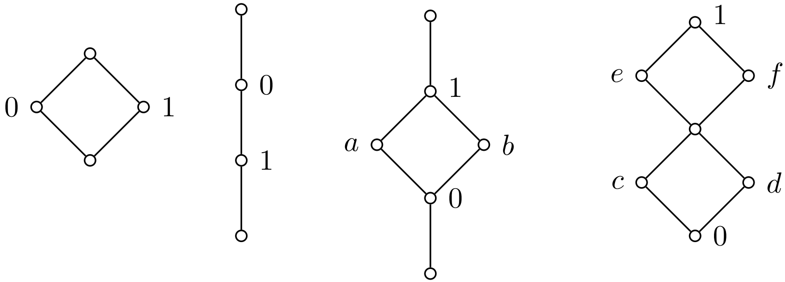
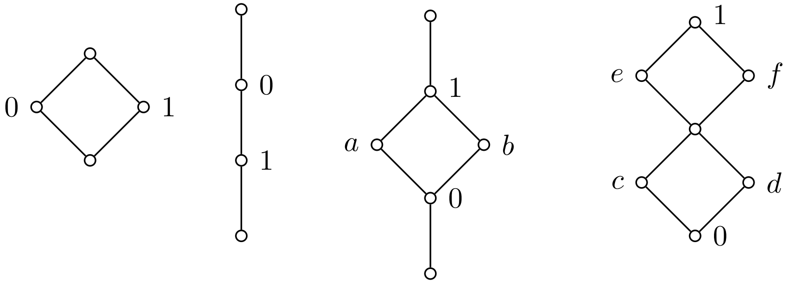
- We have $- \sim a = a = \sim -a$ for all $a \in A$ and $-0 = 1 = \sim 0$.
- The operations $\sim, -, '$ are all dual lattice isomorphisms.
- A qRA (or even an FL-algebra) is cyclic if $\sim a = -a$ for all $a \in A$.
- If $\langle A, \wedge, \vee \rangle$ is distributive and $'$ is complementation, then $\langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0 \rangle$ is a relation algebra.

Examples of quasi relation algebras



$\mathbf{A}_1$ $\mathbf{A}_2$ $\mathbf{A}_3$ $\mathbf{A}_4$

$\mathbf{A}_3$: $-a = \sim a = b$, $-b = \sim b = a$, $a' = a$, $b' = b$



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$\mathbf{A}_4$ is the smallest non-cyclic qRA.

$\mathbf{A}_4$: $\sim: c \to f \to d \to e \to c$
$- : c \to e \to d \to f \to c$
$' : c \to e \to c, d \to f \to d$

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Decidability of quasi relation algebras

Theorem (Galatos, J. 2013)

If $\mathcal{V} = \{\mathbf{A} \in \mathsf{InFL} \mid \mathbf{A} \models \mathcal{E}\}$ for $\mathcal{E}$ a self-dual set of identities and $\mathcal{V}$ is equationally decidable then $\mathcal{V}' = \{\mathbf{A} \in \mathsf{qRA} \mid \mathbf{A} \models \mathcal{E}\}$ is also decidable.

Using results of [Holland, McCleary 1979], [Yetter 1990], [Wille 2005], [Kozak 2011], [Galatos, J. (Res. Frames) 2013]:

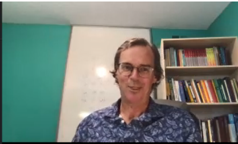
Corollary

*The equational theories of **qRA**, cyclic **qRA**, cyclic distributive **qRA**, commutative **qRA** and the variety of qRAs that have ℓ-group reducts ($= \mathbf{A} \in \mathsf{qRA} \mid A \models x \cdot {\sim}x = 1$) are decidable.*

Theorem (Galatos, J. 2013)

qRA, cyclic qRA and commutative qRA have FMP.

Problem 2: Do (cyclic) **distributive** qRA have the FMP?
Problem 3: Define and investigate **representable qRA**.



Constructing a distributive qRA from a poset: part 1

Let $\mathbf{X} = (X, \leqslant)$ be a poset and E an equivalence relation on X such that $\leqslant \subseteq E$. Define $\preceq$ on E as follows:

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- $R \circ \leqslant = \leqslant \circ R = R$

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Recall that a quasi relation algebra is an algebra of the form

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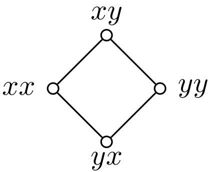
Recall that a quasi relation algebra is an algebra of the form

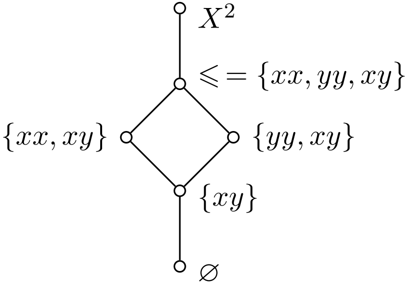
$$\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0, ' \rangle$$ What is 0?

Digression: some examples of the construction so far

$(X, \leqslant)$



$(X^2, \preceq)$



$(\mathsf{Up}\left((X^2, \preceq)\right), \subseteq)$

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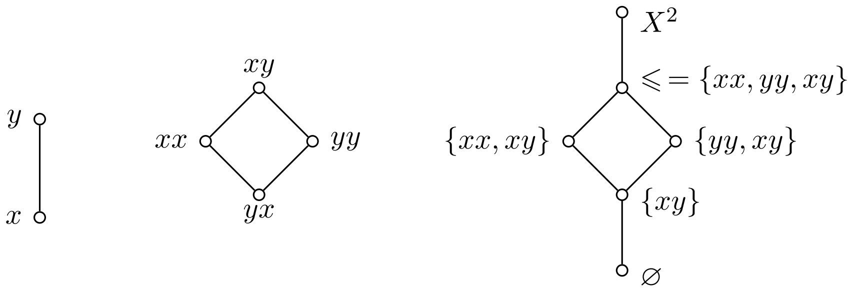
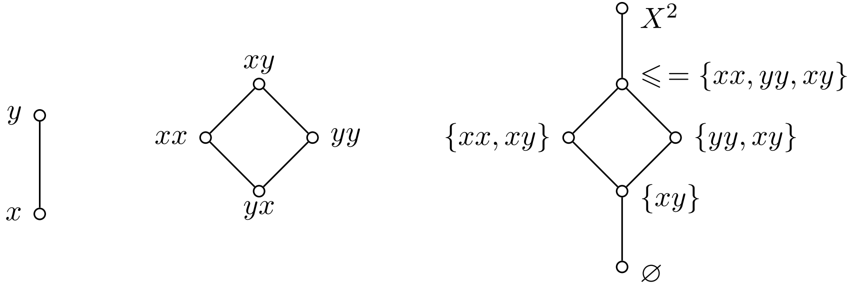


$(X, \leqslant)$ $(X^2, \preceq)$ $(\mathsf{Up}((X^2, \preceq)), \subseteq)$

Since $-1 = 0 = \sim 1$ and $\sim$ and $-$ are dual lattice isomorphisms, the only possibility for 0 is $0 = \{xy\} = (\leqslant^c)^\smile$.

Example of a concrete cyclic distributive qRA



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$$\sim\{xx, xy\} = -\{xx, xy\} = \{yy, xy\} = (\{xx, xy\}^c)^{\smile}$$

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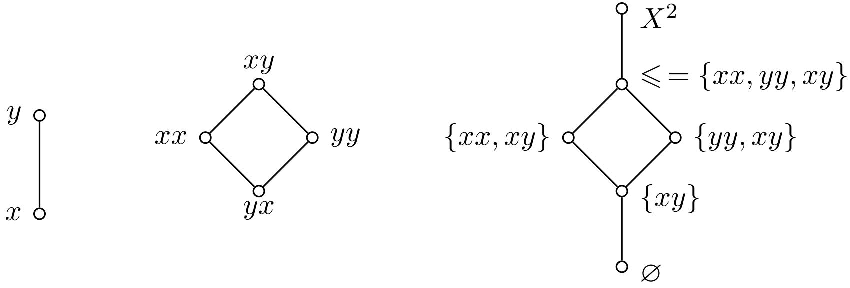

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$$\sim \{xx, xy\} = -\{xx, xy\} = \{yy, xy\} = (\{xx, xy\}^c)^\smile$$
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$$\{xx, xy\}' = \{xx, xy\} = \{xy, yx\} \circ \{xx, xy\}^c \circ \{xy, yx\}$$
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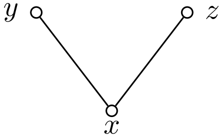
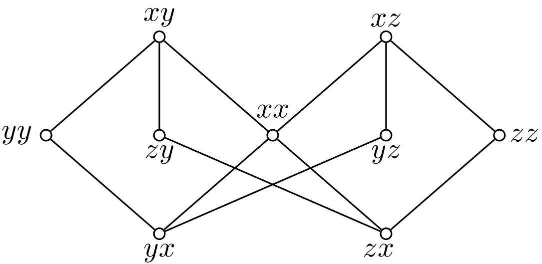
Digression: non-cyclic examples



$(X, \leqslant)$



$(X^2, \preceq)$

- Possibility for 0: $\uparrow yy \cup \uparrow zz = (\leqslant^c)^{\smile} \circ \{xx, yz, zy\}$.

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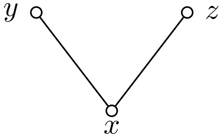

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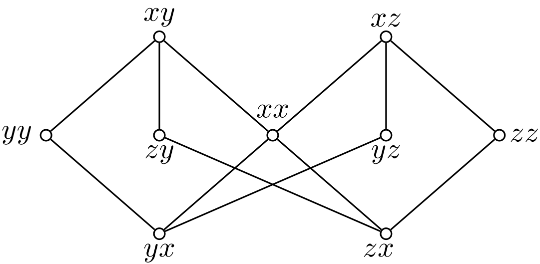

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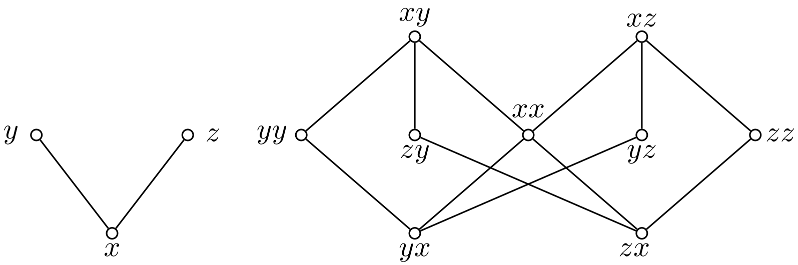

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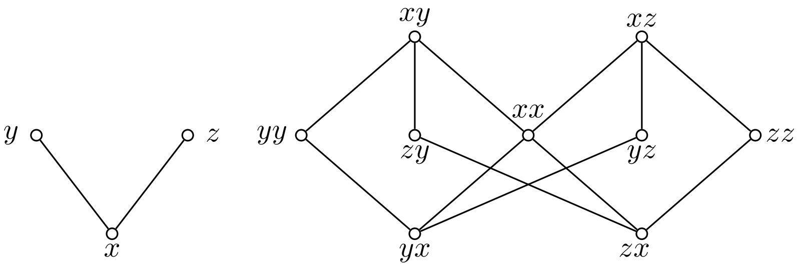

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- In this case (Dp) fails.
- (Dp) fails for all possibilities of $'$ (even if $0 = (\leqslant^c)$).

Constructing a distributive qRA from a poset: part 2

Theorem

Let $\mathbf{X} = (X, \leqslant)$ *be a poset and* E *an equivalence relation on* X *such that* $\leqslant \subseteq E$.

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$$\mathcal{Q}(\mathbf{E}) = \langle \mathsf{Up}(\mathbf{E}), \cap, \cup, \circ, \backslash, /, 1, 0, ' \rangle$$

is a distributive quasi relation algebra.

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$$Q\,(\mathbf{E}) = \left\langle \mathsf{Up}\,(\mathbf{E})\,, \cap, \cup, \circ, \backslash, /, 1, 0,' \right\rangle$$

is a distributive quasi relation algebra. If α *is the identity, then* $Q(\mathbf{E})$ *is a cyclic distributive quasi relation algebra.*

We will refer to the algebra $Q(\mathbf{E})$ as an equivalence distributive quasi relation algebra. The class of equivalence distributive quasi relation algebras will be denoted by **EdqRA**.

Proof of (Di)

Some helpful equivalences:

$$(R^{\smile})^{\mathsf{c}} = (R^{\mathsf{c}})^{\smile} \qquad \beta \circ \alpha^{\smile} = \alpha \circ \beta \qquad \beta = \beta^{\smile}$$

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Let X be a set and $R, S \subseteq X^2$. If $\gamma \colon X \to X$ is an injective function, then the following hold:

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Let $\mathbf{X} = (X, \leqslant)$ be a poset and E an equivalence relation on X such that $\leqslant \subseteq E$. If $\alpha \colon X \to X$ is an order automorphism of $\mathbf{X}$ such that $\alpha \subseteq E$, then $0 = \alpha \circ (\leqslant^c)^{\smile} = (\leqslant^c)^{\smile} \circ \alpha$ iff ${\sim}R = (R^c)^{\smile} \circ \alpha$ and $-R = \alpha \circ (R^c)^{\smile}$ for all $R \in \mathsf{Up}(\mathbf{E})$.

Proof of (Di)

Let $R \in \mathsf{Up}(\mathbf{E})$. Then we have

$$\begin{aligned}
(\sim R)' &= \alpha \circ \beta \circ (\sim R)^c \circ \beta \\
&= \alpha \circ \beta \circ ((R^c)^{\smile} \circ \alpha)^c \circ \beta \\
&= (\alpha \circ \beta \circ (R^c)^{\smile} \circ \alpha \circ \beta)^c \\
&= (\alpha \circ \beta \circ (\alpha^{\smile} \circ R^c)^{\smile} \circ \beta)^c \\
&= (\alpha \circ (\beta^{\smile} \circ \alpha^{\smile} \circ R^c \circ \beta^{\smile})^{\smile})^c \\
&= (\alpha \circ (\beta \circ \alpha^{\smile} \circ R^c \circ \beta)^{\smile})^c \\
&= \alpha \circ ((\beta \circ \alpha^{\smile} \circ R^c \circ \beta)^{\smile})^c \\
&= \alpha \circ ((\alpha \circ \beta \circ R^c \circ \beta)^{\smile})^c \\
&= \alpha \circ ((R')^c)^{\smile} \\
&= -(R')
\end{aligned}$$

Full distributive qRAs

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We will refer to this algebra as a full distributive quasi relation algebra and denote the class of full distributive quasi relation algebras by **FdqRA**.

Representable distributive qRAs

Theorem

$$\mathbb{ISP}\left(\mathbf{FdqRA}\right) = \mathbb{IS}\left(\mathbf{EdqRA}\right)$$

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Definition

A distributive quasi relation algebra $\mathbf{A}$ is *representable* if

$$\mathbf{A} \in \mathbb{ISP}\left(\textbf{FdqRA}\right)$$

or, equivalently,

$$\mathbf{A} \in \mathbb{IS}\left(\textbf{EdqRA}\right).$$

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- Do the answers to the above change when we restrict to representable cyclic distributive quasi reltion algebras?
- Can we use TiRS graphs (or some other class of dual structures) to define representability for (cyclic) quasi relation algebras?

Thank you!

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