Explicit results on the bound of Siegel zeros for quadratic fields

F. Brice Razakarinoro

Stellenbosch
UNIVERSITY
IYUNIVESITHI
UNIVERSITEIT

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Motivation

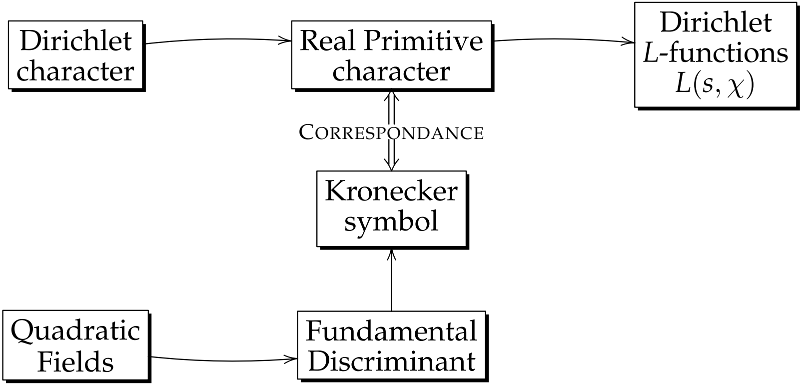
- Many results in Number Theory use informations on prime numbers.
- The study of primes requires properties of certain special functions such as $\zeta(s)$, $L(s, \chi)$, $\ldots$
- In particular, we are interested in the **locations of the zeros** of the L-functions attached to a Dirichlet character χ ... existence of hypothetical zero!

Aim: Find an explicit bound of the hypothetical zero, i.e. Siegel zero, if it exists.

Overview:

1. Preliminaries
2. Standard techniques
3. Some explicit results

Diagram of the background materials



Primitive characters

Let q be a positive integer.
• A **Dirichlet character** χ modulo q, denoted by $\chi(\mathrm{mod}\ q)$, is a function from $\mathbb{Z}$ to $\mathbb{C}^*$ such that:

- $\chi(n) = \chi(n+q)$ for every n,
- $\chi(n) = 0$ if $\gcd(n,q) > 1$,
- $\chi(mn) = \chi(m)\chi(n)$ for every m and n.

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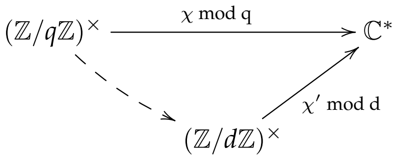
• A Dirichlet character χ (mod q) is called **primitive** if there is no divisor d of q such that



Quadratic number fields

• An **(algebraic) number field** K is a *finite degree* field extension of the $\mathbb{Q}$. We usually denote by $[K:\mathbb{Q}]$ the degree of the extension $K/\mathbb{Q}$.

• When $[K:\mathbb{Q}] = 2$, we say that K is a **quadratic number field**.

• More precisely, quadratic fields are of the form $\mathbb{Q}(\sqrt{D})$ for some squarefree integer D; and if

$$\begin{cases} D < 0 & \text{we say } K \text{ is an } \textbf{imaginary} \text{ quadratic field,} \\ D > 0 & \text{we say } K \text{ is a } \textbf{real} \text{ quadratic field.} \end{cases}$$

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• For any quadratic field K, we define the **discriminant** of K as an integer Δ_K such that

$$\Delta_K = \begin{cases} D & \text{if } D \equiv 1 \bmod 4, \\ 4D & \text{if } D \not\equiv 1 \bmod 4. \end{cases}$$

Real primitive characters

• A **fundamental discriminant** is an integer, say D, such that there exists $K/\mathbb{Q}$ a quadratic field with $D = \Delta_K$ the discriminant of K.

• The **Kronecker symbol** $\left(\frac{D}{n}\right)$ is a generalization of the Jacobi symbol. In particular, we have

$$\left(\frac{D}{-1}\right) = \operatorname{sign}(D). \tag{1}$$

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Theorem: $''(\frac{D}{n}) \Longleftrightarrow \chi$ real prim. $''$

If D is a fundamental discriminant, the Kronecker symbol $(\frac{D}{n})$ defines a **real primitive** character modulo $|D|$.
Conversely, if χ is a real primitive character modulo q then $D = \chi(-1)q$ is a **fundamental discriminant** and $\chi(n) = (\frac{D}{n})$.

Dirichlet L-functions and its zeros

• From (1), real primitive characters depend on $\chi(-1)$. We say that χ is **odd** if $\chi(-1) = -1$, and χ is **even** if $\chi(-1) = 1$.

• Now, we define the **Dirichlet L-function** attached to the character $\chi(n) = (\frac{D}{n})$ as

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad (Re(s) > 1).$$

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- if χ is odd, then $s = -1, -3, -5, \dots$ are zeros,
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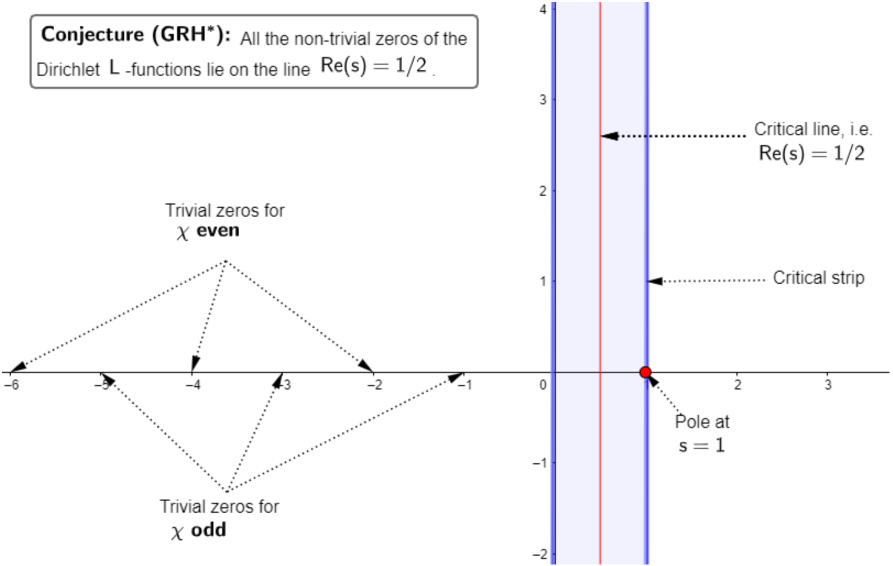
• For $Re(s) > 0$, $L(s, \chi)$ has **no** zeros with $Re(s) > 1$. So, the remaining zeros, which are called **non-trivial** zeros, lie in the region

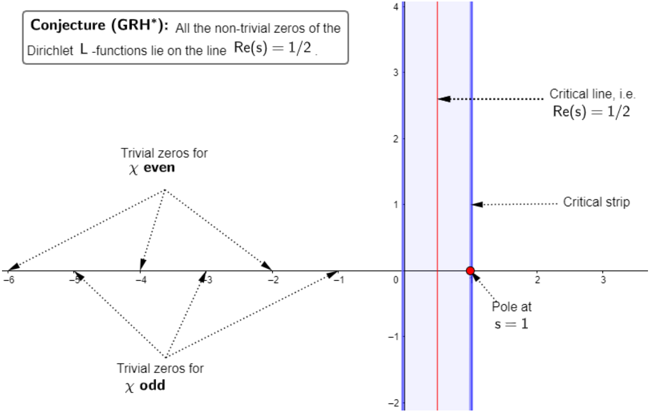
$$\{s \in \mathbb{C} \mid 0 < Re(s) < 1\}.$$

This region is called the **critical strip**. . . GRH!



***GRH**: Generalized Riemann Hypothesis

The zeros (2/2)



Theorem: (Platt, 2016)

GRH holds for Dirichlet L-functions of primitive characters modulo $q \leq 400\,000$ and to height $\max\left(\frac{10^8}{q}, \frac{A\cdot 10^7}{q}\right)$, with $A = 7.5$ if the character is even, and $A = 3.5$ if it is odd.

Zero-free regions (1/2)

Write $s = \sigma + it$. Gronwall (1913), Landau (1918) and Titchmarsh (1933) independently showed the following result.

Theorem:

There exists $c > 0$ such that: if $\chi (\mathrm{mod}\ q)$ is a **complex** character, then $L(s, \chi)$ has no zeros in the region defined by

$$\sigma \geq \begin{cases} 1 - \frac{c}{\log q|t|} & \text{if } |t| \geq 1, \\ 1 - \frac{c}{\log q} & \text{if } |t| \leq 1. \end{cases}$$

If $\chi (\mathrm{mod}\ q)$ is a **real** nonprincipal character, the only possible zero of $L(s, \chi)$ in this region is a single (simple) real zero.

• This hypothetical real zero is called **Siegel zero** or **Landau-Siegel zero**. If it exists, it is known to be **very close** to 1.

Zero-free regions (2/2)

The latest known **explicit zero-free regions** for Dirichlet L-functions is due to H. Kadiri (2018).

Theorem: (Kadiri, 2018)

Let q be an interger with $3 \leq q \leq 400\,000$ and χ a non-principal primitive character modulo q. Then the Dirichlet L-function $L(s, \chi)$ does not vanish in the region

$$Re(s) \geq 1 - \frac{1}{5.60\ \log\left(q \max(1, |Im(s)|)\right)}.$$

• This improves a result of McCurley (1984) where 9.65 was shown to be an admissible constant.

Overview:

Study of β: standard method

Denote by β the Siegel zero, if it exists. To study β, we often look at the distance of β from $1 \Rightarrow$ **bound** the quantity $1 - \beta$.

- A standard way to do this is by using the **Mean Value Theorem** for $L(s, \chi)$. That is,

$$L(1, \chi) = L(1, \chi) - L(\beta, \chi) = L'(\sigma, \chi)(1 - \beta),$$

for some $\sigma \in (\beta, 1)$. Hence,

$$1 - \beta = \frac{L(1, \chi)}{|L'(s, \chi)|}. \tag{2}$$

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By estimating the RHS, we end up with a bound of the form

$$\beta \leq 1 - \frac{\lambda_1}{\sqrt{q}\log^2 q}, \tag{3}$$

and λ_1 can be computed explicitly.

Goldfeld-Schinzel's approach

- Another approach considers certain **integral** involving $\zeta_{\mathbb{Q}\sqrt{d}}(s)$. Following the method of Goldfeld and Schinzel (1975), one needs to study the **reciprocal sums of ideals norms**

$$\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})}, \tag{4}$$

for any $\mathfrak{a} \subset O_K$, where K is a quadratic field. This is because when estimating the considered integral, we obtain

$$1 - \beta \sim \frac{6}{\pi^2} \frac{L(1,\chi)}{\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})}}. \tag{5}$$

At the end, we get

$$\beta \leq \begin{cases} 1 - \frac{\lambda_2}{\sqrt{q}} & \text{for odd } \chi, \\ 1 - \frac{\lambda_3 \log d}{\sqrt{d}} & \text{for even } \chi, \end{cases} \tag{6}$$

where λ_2 and λ_3 are effectively computable constants.

Overview:

Some theorems (1/3)

Theorem: (Page, 1935)

If χ is a real non-principal character modulo q, and if β_1 and β_2 are real zeros of $L(s, \chi)$ then there is a constant $c > 0$ such that

$$\min(\beta_1, \beta_2) \leq 1 - \frac{c}{\log q}.$$

Recently, an explicit value of c was given. We have

Theorem: (Morrill & Trudgian, 2019)

If χ mod q is a real non-principal character, with $q \geq 3$, and if β_1 and β_2 are real zeros of $L(s, \chi)$, then

$$\min(\beta_1, \beta_2) \leq 1 - \frac{0.933}{\log q}.$$

Some theorems (2/3)

Theorem: (Liu & Wang, 2002)

Let χ be a (nonprincipal) real primitive character modulo $q \geq 987$. If there exists $\beta > 0$ such that $L(\beta, \chi) = 0$, then we have

$$\beta \leq 1 - \frac{\pi}{0.4923\sqrt{q}\log^2 q}.$$

Theorem: (Ford, Luca & Moree , 2014)

Let $q \geq 10000$ be prime and let χ be the quadratic character modulo q. If $L(\beta_0, \chi) = 0$, then

$$\beta_0 \geq 1 - \frac{3.125\,\min(2\pi, \frac{1}{2}\log q)}{\sqrt{q}\log^2 q}.$$

Some theorems (3/3)

Theorem: (Bennett et al., 2018)

Let $q \geq 3$ be an integer and let χ be a quadratic character modulo q. If $\beta > 0$ is a real number for which $L(\beta, \chi) = 0$, then

$$\beta \leq 1 - \frac{40}{\sqrt{q}\log^2 q}.$$

Theorem: (Bordignon, 2019 & 2020)

Assume χ is a non-principal real character and $q > 4.10^5$. We have

$$\beta \leq \begin{cases} 1 - \frac{800}{\sqrt{q}\log^2 q} & \text{for } \chi \text{ odd} \\ 1 - \frac{100}{\sqrt{q}\log^2 q} & \text{for } \chi \text{ even.} \end{cases}$$

Bound of different form

Haneke (1973), Pintz (1976) and Goldfeld-Schnizel (1975) gave bounds of the form different from the previous results. In particular,

Theorem: (Goldfeld & Schinzel, 1975)

Let d be a fundamental discriminant and let $L(s, \chi)$ be the L-function associated to $\chi(n) = (\frac{d}{n})$. If there is $\beta > 0$ such that $L(\beta, \chi) = 0$ then there exist an effectively computable constant λ_1 and λ_2 such that

$$\beta \leq \begin{cases} 1 - \frac{\lambda_1}{\sqrt{|d|}} & \text{if } d < 0, \\ 1 - \frac{\lambda_2 \log d}{\sqrt{d}} & \text{if } d > 0. \end{cases}$$

• If one could provide an explicit value for the above constant, this results is far better than any known explicit results in the literature.

An explicit bound of different form

• During his MSc project, using Goldfeld-Schnizel's approach with **known computational results** and **analytic tools**, the author found that $\lambda_1 = 1.151$ when $d < 0$. By sharpening the method used, we obtained

Theorem: (Naina & B.)

Let $d < 0$ be a fundamental discriminant and $L(s, \chi)$ be the L-function attached to the character $\chi(n) = (\frac{d}{n})$. If there exists $\beta > 0$ such that $L(\beta, \chi) = 0$ then

$$\beta \leq 1 - \frac{6.5}{\sqrt{|d|}}.$$

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$$\beta \leq 1 - \frac{6.5}{\sqrt{|d|}}.$$

• When $d > 0$, we can also achieve λ_2 by studying in more details the **structure of real quadratic fields** $\mathbb{Q}(\sqrt{d})$. So far, we find that $\lambda_2 = 0.097$, but this is still in progress ...

Thank you very much!