The nonnegative inverse eigenvalue problem with prescribed zero patterns in low dimensions

André Ran,
in collaboration with Emily Teng

VRIJE UNIVERSITEIT AMSTERDAM

Faculty of Science

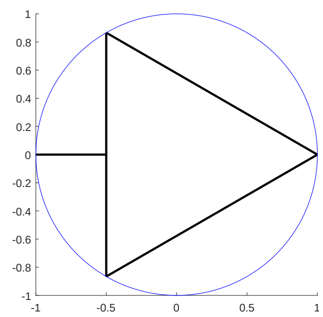
NWU

Setting

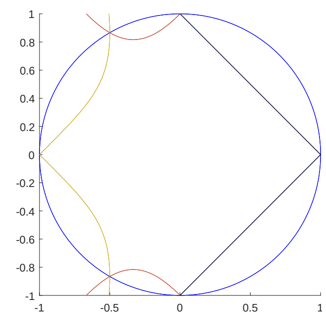
A an $n \times n$ matrix with only nonnegative entries.
Without loss of generality row-stochastic (i.e., row sums are all one).

Perron-Frobenius theorem with all its ramifications gives information on where the eigenvalues can be:

- 1 is an eigenvalue, and is the largest eigenvalue in modulus,
- there are restrictions on the eigenvalues on the unit circle,
- all eigenvalues are in the so-called Karpelevich region.



$n = 3$



$n = 4$

Problem statement: background

Given a set of n complex numbers, $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ give necessary and sufficient conditions for the existence of an $n \times n$ nonnegative matrix A such that the set of eigenvalues of A is the set Λ.

Obvious conditions: Perron-Frobenius needs to be satisfied, the set Λ must be in the Karpelevich region, and must be symmetric with respect to the real line.

Very hard problem: solution exists only for $n = 3$ and $n = 4$ (despite decades of work).

Many extra necessary conditions are known by now.

Very nice review papers, among them one by C.R. Johnson et al. (2018).

Interesting observation: for $n = 5$ with the additional condition that the trace is zero (so the diagonal is zero) there are necessary and sufficient conditions.

Problem statement

Let $\Lambda = \{1, \lambda_2, \lambda_3\}$ be a set of three numbers in the Karpelevich region for $n = 3$,
with the obvious condition $\lambda_3 = \overline{\lambda_2}$ in case $\lambda_{2,3} \notin \mathbb{R}$,
and take $\lambda_3 \leq \lambda_2$ in case $\lambda_{2,3} \in \mathbb{R}$.

Let also a zero pattern in 3×3 matrices be given. Give necessary and sufficient conditions for there to exist a nonnegative matrix A with the prescribed zero pattern and such that $\sigma(A) = \Lambda$.

In other words, what extra restrictions does a zero pattern impose on the location of the eigenvalues of a nonnegative matrix?

In this lecture only a few interesting cases, but a complete description for all possible zero patterns in the 3×3 case is available.

First observations

Without loss of generality A is row stochastic, that is, $A\mathbf{1} = \mathbf{1}$.

A is called *reducible* if there is a permutation of rows and columns such that $A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$.
Otherwise A is called *irreducible*.
The reducible cases are simple: because of row stochasticity there are three real eigenvalues, one equal to 1, one positive, and the third one may be positive or negative, depending on the sign of the determinant.

So we focus on irreducible and row stochastic matrices.

The problem is not completely invariant under similarity with permutation matrices: $A \to P^{-1}AP$ with P a permutation matrix, because that changes a zero pattern into an equivalent one.

Zeroes on the diagonal

One zero on the diagonal.
Use: $P^{-1}AP$ has same eigenvalues as A for any permutation P to place that zero in $(1,1)$ position.

$$A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & \gamma & 1-\beta-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$$ nonnegative matrix. $\sigma(A) = \{1, \lambda_2, \lambda_3\}$.

Then

i. If $\lambda_{2,3} = a \pm ib$ with $b \neq 0$ then $a + b^2 < \frac{1}{4}$.

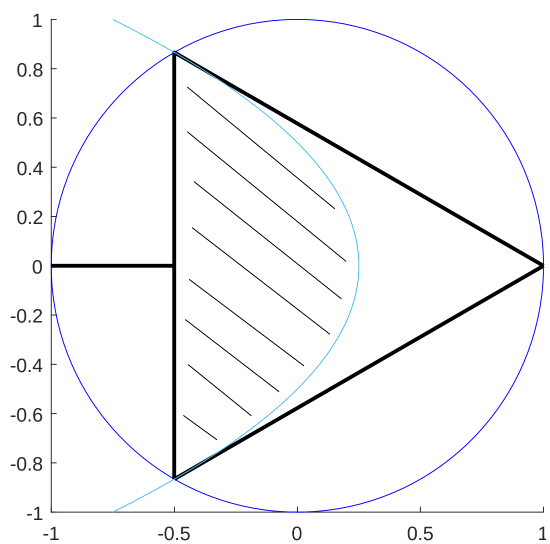
ii. If $\lambda_3 \leq \lambda_2$ are real, then $-1 < \lambda_2 + \lambda_3 < 1$, and the point (λ_2, λ_3) lies in the region in the real plane bounded by the lines $\lambda_2 = \lambda_3$, $\lambda_2 + \lambda_3 = -1$, $\lambda_3 = -1$, $\lambda_2 = 1$ and the parabola $1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2 = 0$.

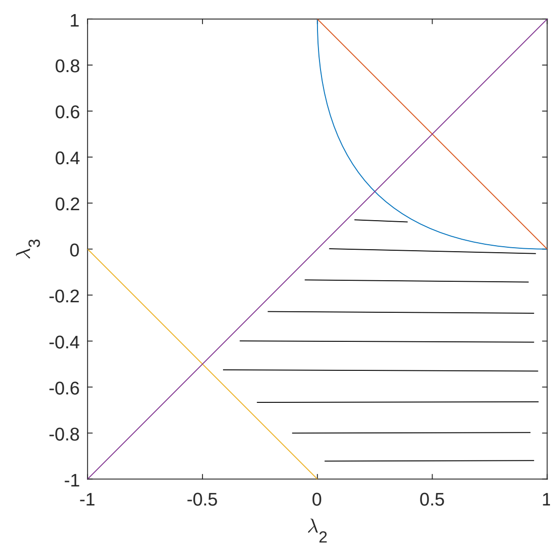
Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.

Zeroes on the diagonal

If $\lambda_{2,3} = a \pm ib$ with $b \neq 0$ then $a + b^2 < \frac{1}{4}$ (figure left).

If $\lambda_3 \leq \lambda_2$ are real, then $-1 < \lambda_2 + \lambda_3 < 1$, and the point (λ_2, λ_3) lies in the region in the real plane bounded by the lines $\lambda_2 = \lambda_3$, $\lambda_2 + \lambda_3 = -1$, $\lambda_3 = -1$, $\lambda_2 = 1$ and the parabola $1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2 = 0$ (figure right).





Zeroes on the diagonal

Two zeroes on the diagonal.

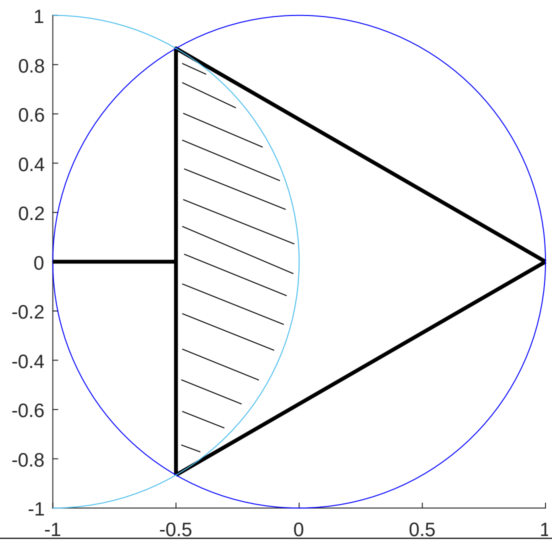
$$A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & 0 & 1-\beta \\ \gamma & \delta & 1-\gamma-\delta \end{bmatrix}$$ nonnegative matrix. $\sigma(A) = \{1, \lambda_2, \lambda_3\}$.

Then

i. If $\lambda_{2,3} = a \pm ib$ with $b \neq 0$ then $(a+1)^2 + b^2 < 1$.

ii. If $\lambda_3 \leq \lambda_2$ are real, then $-1 < \lambda_2 + \lambda_3 < 0$.

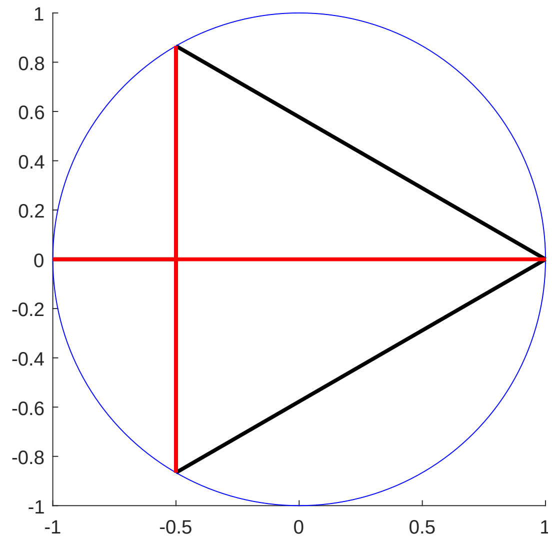
Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.



Zeroes on the diagonal

Three zeroes on the diagonal and no other zero entries.
Trace is zero, so sum of the eigenvalues is zero.

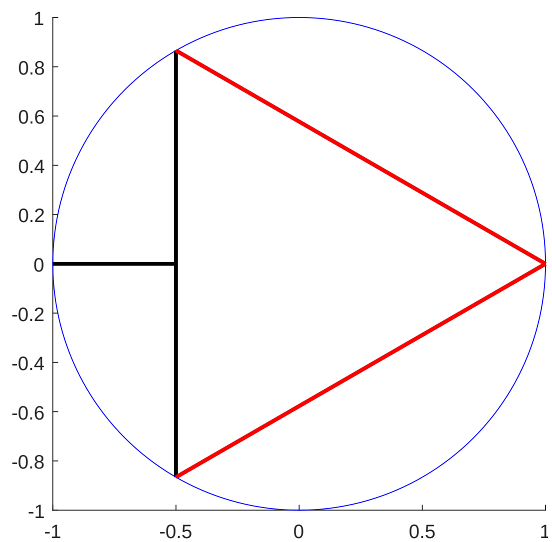
Then
either $\sigma(A) = \{1, -\frac{1}{2} + bi, -\frac{1}{2} - bi\}$ with $0 < b < \frac{\sqrt{3}}{2}$,
or $\sigma(A) = \{1, -\frac{1}{2} + a, -\frac{1}{2} - a\}$ with $0 \leq a < \frac{1}{2}$.



Boundary of the Karpelevich region forces a pattern

Suppose A is a row stochastic matrix with eigenvalues 1 and $\lambda_{1,2} = a \pm ib$ with $b = \frac{\sqrt{3}}{3}(1-a)$
(so on the line segments connecting 1 and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$).
Then there is a permutation matrix P and $0 \le t \le 1$ such that

$$A = P \begin{bmatrix} t & 0 & 1-t \\ 1-t & t & 0 \\ 0 & 1-t & t \end{bmatrix} P^{-1}.$$



Circulant zero pattern

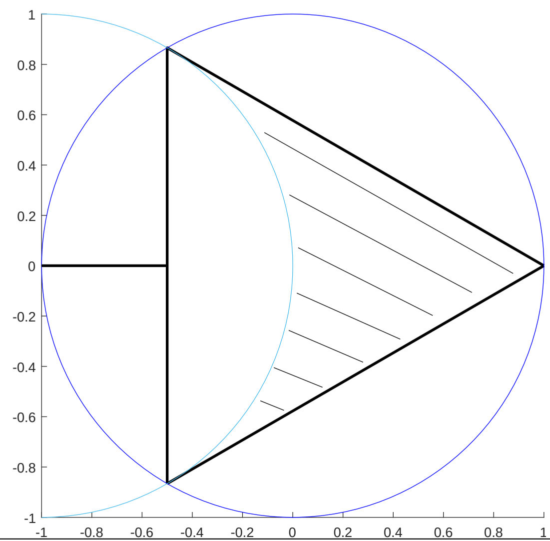
$$A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 1-\beta & \beta & 0 \\ 0 & 1-\gamma & \gamma \end{bmatrix}, \text{ with } \sigma(A) = \{1, \lambda_2, \lambda_3\}.$$

Then

i. if λ_2, λ_3 are real they are both positive,

ii. if $\lambda_{2,3} = a \pm bi$ are non-real then we have $(a+1)^2 + b^2 > 1$ and $|b| \leq \sqrt{3}(1-a)$.

Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.



Circulant zero pattern plus one extra zero on diagonal

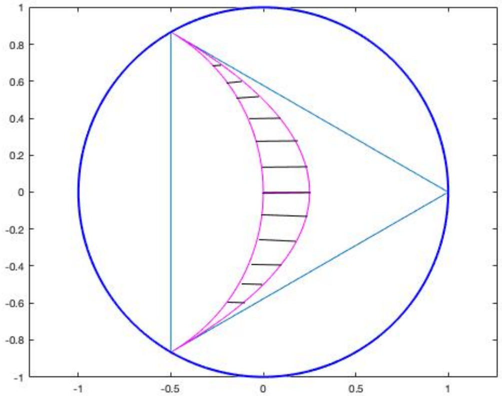
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1-\alpha \\ 1-\beta & 0 & \beta \end{bmatrix}, \text{ with } \sigma(A) = \{1, \lambda_2, \lambda_3\}.$$

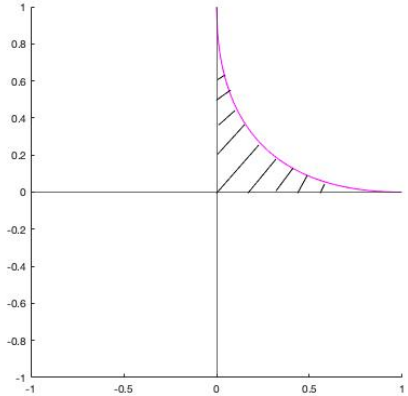
Then

i. If $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, then $a \in (-\frac{1}{2}, \frac{1}{4})$ with $(a+1)^2 + b^2 > 1$ and $a + b^2 < \frac{1}{4}$;

ii. If $\lambda_2 \geq \lambda_3$ are real, then $\lambda_2, \lambda_3 > 0$ and $\Delta := 1 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_2 + \lambda_3 + \lambda_2\lambda_3) > 0$.

Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.

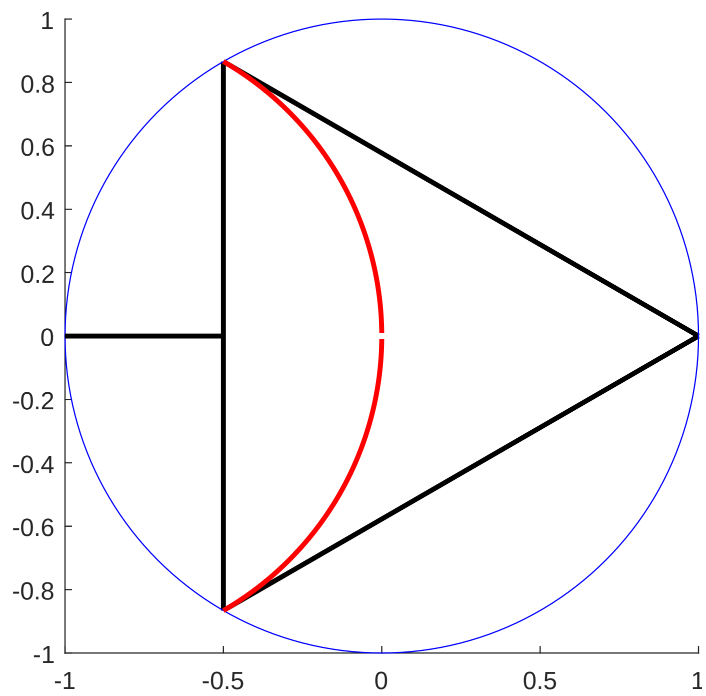




Circulant zero pattern plus two extra zeroes on diagonal

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-\alpha & 0 & \alpha \end{bmatrix}, \quad 0 < \alpha < 1, \text{ with } \sigma(A) = \{1, \lambda_2, \lambda_3\}.$$

Then $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, and $(a+1)^2 + b^2 = 1$.



An example in dimension four

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & \beta & 1-\beta & 0 \\ 0 & 0 & \gamma & 1-\gamma \\ 1-\delta & 0 & 0 & \delta \end{bmatrix}.$$

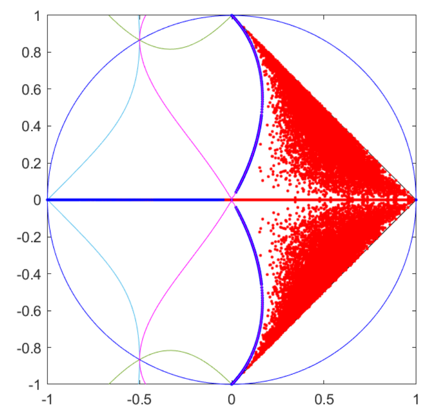
Introduce the region

$$R = \{z = a + bi \mid z \in K, a > 0, (b^2 + a^2 + a)^2 + 2a^2 - b^2 > 0\}.$$

We conjecture the following.

Conjecture *For any irreducible nonnegative matrix A of this form the non-real eigenvalues of A are in the region R.*

The region R and some support for the conjecture



Spectrum of 10^4 random matrices of the form above.

Eigenvalues of matrices of the form $A(\alpha) = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

are on the curve $(b^2 + a^2 + a)^2 + 2a^2 - b^2 = 0$.

The boundary curve of the region R, indicated in dark blue.

The region R and some support for the conjecture

$$A(t) = tA(\alpha) + (1-t)I = \begin{bmatrix} t\alpha + (1-t) & t(1-\alpha) & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 0 & 1-t & t \\ t & 0 & 0 & 1-t \end{bmatrix}.$$

Eigenvalues are on the straight line segments connecting points on the boundary curve with the point 1.

That fills out the region R.

Some comments on proofs and techniques

Usually the hard part is to prove that a given set of numbers in the region is realised as the set of eigenvalues of a nonnegative matrix with the prescribed zero pattern.

Sometimes this can be done by explicit construction of a nonnegative matrix with more zeroes, and then using the implicit function theorem to prove that it also works with fewer zeroes.

Important point: λ_2, λ_3 are fully determined by $\det(A)$ and trace (A).

Conclusions

Almost all zero patterns in the three dimensional case impose extra conditions.

For each given zero pattern in dimension three we can describe what these extra conditions are.

In dimension four almost everything is still open.

Obviously, the nonnegative inverse eigenvalue problem is still very much on the wish list of many people, but also unlikely to be solved in the near future in all its generality.

Thank you for your attention