Constructing Fischer-Clifford matrices of a finite extension group from its factor group

Abraham Love Prins

Department of Mathematics and Applied Mathematics, Nelson Mandela University, Port Elizabeth,
email: abraham.prinsab@mandela.ac.za

6 December 2022

Introduction

- Let N be a normal subgroup of a finite group F. Then it is well-known that the ordinary irreducible characters $\mathrm{Irr}(Q)$ of the quotient group $Q = \frac{F}{N}$ can be lifted to F, where the set $\mathrm{Irr}(Q)$ is identified with $\chi_i \in \mathrm{Irr}(F)$ such that $N \leq \ker(\chi_i)$.
- In this presentation, we will use an analogous process to "lift" the so-called Fischer-Clifford matrices $\widehat{M(g_i)}$ of the quotient group $\overline{Q} = \frac{\overline{G}}{K} \cong P_1.G$ to the corresponding Fischer-Clifford matrices $M(g_i)$ of a finite extension group $\overline{G} = P.G$, where $K \lhd \overline{G}$ is a non-trivial characteristic subgroup of the p-group P.
- Hence we can add the necessary rows and columns to the matrices $\widehat{M(g_i)}$ to completely construct the matrices $M(g_i)$ and ordinary character table of $\overline{G}$.

Fischer-Clifford matrices

Let $\overline{G} = N.G$ be an extension of N by G, where $N \triangleleft \overline{G}$ and $\frac{\overline{G}}{N} \cong G$. Also, let $\theta_1 = 1_N, \theta_2, \cdots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$, where $\overline{H_i} = \left\{x \in \overline{G} | \theta_i^x = \theta_i\right\}$ is the corresponding inertia group of $\theta_i \in \mathrm{Irr}(N)$ in $\overline{G}$ for $1 \leq i \leq t$. In addition, we have the inertia factor $H_i = \frac{\overline{H_i}}{N}$ corresponding to $\overline{H_i}$.

It follows from Gallagher [11] that

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t} \{(\psi_i \overline{\beta})^{\overline{G}} | \beta \in \mathrm{IrrProj}(H_i), \text{ with factor set } \alpha_i^{-1}\},$$

where $\psi_i \overline{\beta}$ is equivalent to an ordinary irreducible character $\chi \in \mathrm{Irr}(\overline{H}_i)$ of $\overline{H}_i$ such that $< \chi_N, \theta_i >_N \neq 0$. Moreover ψ_i is a fixed projective character of $\overline{H_i}$ with factor set $\overline{\alpha}_i$ and is an extension of θ_i to $\overline{H_i}$, i.e. $(\psi_i)_N = \theta_i$. Since $\overline{\alpha_i}$ is constant on cosets of N in $\overline{H_i}$ it can be identified as a factor set α_i of the inertia factor H_i and is defined as $\alpha_i(Nv, Nw) = \overline{\alpha}_i(v, w)$ for $v, w \in \overline{H_i}$.

Fischer-Clifford matrices

Let $X(g) = \{x_1 = \overline{g}, x_2, \cdots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset $N\overline{g}$, where $\overline{g}$ be a lifting of $g \in G$ under the natural homomorphism $\overline{G} \longrightarrow G$. Note that g is identified with the coset $N\overline{g}$.
$y_1, y_2, .., y_r$ to be representatives of the α_i^{-1}-regular conjugacy classes of elements of H_i that fuse to $[g]$ in G. We define

$$R(g) = \{(i, y_k) \,|\, 1 \leq i \leq t, H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\},$$

where y_k are representatives of the α_i^{-1}-regular classes of H_i that fuse into the class $[g]$ of G.

We define $y_{l_k} \in \overline{H_i}$ such that y_{l_k} ranges over all representatives of the conjugacy classes of elements of $\overline{H_i}$ which map to y_k under the homomorphism $\overline{H_i} \longrightarrow H_i$ whose kernel is N.

Lemma 1

With notation as above,

$$(\psi_i\overline{\beta})^{\overline{G}}(x_j) = \sum_{y_k:(i,y_k)\in R(g)} \beta(y_k) \sum_{l}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})$$

Proof.

See [22] □

Fischer-Clifford matrices (cont)

Then the Fischer matrix $M(g) = \left(a^j_{(i,y_k)}\right)$ is defined as

$$\left(a^j_{(i,y_k)}\right) = \left(\sum_l{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})\right),$$

with columns indexed by $X(g)$ and rows indexed by $R(g)$ and where $\sum_l'$ is the summation over all l for which $y_{l_k} \sim x_j$ in $\overline{G}$.

Fischer-Clifford matrices (cont)

The Fischer $M(g)$ (see Figure 1) is partitioned row-wise into blocks, where each block corresponds to an inertia group $\overline{H}_i$. We write $|C_{\overline{G}}(x_j)|$, for each $x_j \in X(g)$, at the top of the columns of $M(g)$ and at the bottom we write $m_j \in \mathbb{N}$, where we define $m_j = [C_{\overline{g}}{:}C_{\overline{G}}(x_j)]{=}|N|\frac{|C_G(g)|}{|C_{\overline{G}}(x_j)|}$ and $C_{\overline{g}} = \{x \in \overline{G} | x(N\overline{g}) = (N\overline{g})x\}$. On the left of each row we write $|C_{H_i}(y_k)|$, where the α_i^{-1}-regular class $[y_k]$ fuses into the class $[g]$ of G.

Fischer-Clifford matrices (cont)

Figure 1: The Fischer Matrix $M(g)$

$$
M(g)=\begin{array}{c} \\ |C_G(g)| \\ |C_{H_2}(y_1)| \\ |C_{H_2}(y_2)| \\ \vdots \\ |C_{H_i}(y_1)| \\ |C_{H_i}(y_2)| \\ \vdots \\ |C_{H_t}(y_1)| \\ |C_{H_t}(y_2)| \\ \vdots \\ \\ \end{array}
\begin{array}{cccc}
|C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \\
\hline
a^1_{(1,g)} & a^2_{(1,g)} & \cdots & a^{c(g)}_{(1,g)} \\
\hline
a^1_{(2,y_1)} & a^2_{(2,y_1)} & \cdots & a^{c(g)}_{(2,y_1)} \\
a^1_{(2,y_2)} & a^2_{(2,y_2)} & \cdots & a^{c(g)}_{(2,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
a^1_{(i,y_1)} & a^2_{(i,y_1)} & \cdots & a^{c(g)}_{(i,y_1)} \\
a^1_{(i,y_2)} & a^2_{(i,y_2)} & \cdots & a^{c(g)}_{(i,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
a^1_{(t,y_1)} & a^2_{(t,y_1)} & \cdots & a^{c(g)}_{(t,y_1)} \\
a^1_{(t,y_2)} & a^2_{(t,y_2)} & \cdots & a^{c(g)}_{(t,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
m_1 & m_2 & \cdots & m_{c(g)}
\end{array}
$$

In practice we will never compute the y_{l_k} or the ordinary irreducible character tables of the inertia subgroups $\overline{H}_i$ since the ordinary irreducible characters of the $\overline{H}_i$ are in general much larger and more complicated to compute than the one for $\overline{G}$. Instead of using formal definition, the below arithmetical properties of $M(g)$ are used to compute the entries of $M(g)$[15].

(a) $a^j_{(1,g)} = 1$ for all $j = \{1, 2, .., c(g)\}$.

(b)$|X(g)| = |R(g)|$.

(c) $\sum_{j=1}^{c(g)} m_j \, a^j_{(i,y_k)} \, \overline{a^j_{(i',y'_k)}} = \delta_{(i,y_k),(i',y'_k)} \frac{|C_G(g)|}{|C_{H_i}(y_k)|} |N|$.

(d) $\sum_{(i,y_k)\in R(g)} a^j_{(i,y_k)} \, \overline{a^{j'}_{(i,y_k)}} \, |C_{H_i}(y_k)| = \delta_{jj'} \, |C_{\overline{G}}(x_j)|$.

If N is elementary abelian, then we obtain the following additional properties of $M(g)$:

(f) $a^1_{(i,y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$.

(g) $|a^1_{(i,y_k)}| \geq |a^j_{(i,y_k)}|$.

(h) $a^j_{(i,y_k)} \equiv a^1_{(i,y_k)} (\text{mod } p)$, if $|N| = p^n$, for p a prime and $n \in \mathbb{N}$

Fischer-Clifford matrices (cont)

The partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \cdots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)\, M_1(g) \\ C_2(g)\, M_2(g) \\ \vdots \\ C_t(g)\, M_t(g) \end{bmatrix}$$

where the Fischer matrix $M(g)$ (see Figure 1) is divided into blocks $M_i(g)$ with each block corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ is the partial character table of H_i with factor set α_i^{-1} consisting of the columns corresponding to the α_i^{-1}-regular classes that fuse into $[g]$ in G. We obtain the characters of $\overline{G}$ by multiplying the relevant columns of the projective characters of H_i with factor set α_i^{-1} by the rows of $M(g)$. We can also observe that

$$|\mathrm{Irr}(\overline{G})| = \sum_{i=1}^{t} |\mathrm{IrrProj}(H_i, \alpha_i^{-1})|.$$

On the conjugacy classes of $\overline{G}$, $\overline{Q}$ and G

Let $\overline{G} = P.G$ be a finite extension with $P \lhd \overline{G}$ a p-group. If $K \lhd \overline{G}$ is a non-trivial characteristic subgroup of P then we have the structures $\overline{G} = K.\overline{Q}$ and $\overline{Q} = \frac{\overline{G}}{K} \cong P_1.G$ where $P_1 \cong \frac{P}{K}$. The commutative diagram, depicted as Figure 2, is associated with the structures $\overline{G}$ and $\overline{Q}$, where η_1, η_2 and $\eta = \eta_2 \circ \eta_1$ are the natural homomorphisms from $\overline{G}$ onto $\overline{Q}$, $\overline{Q}$ onto G and $\overline{G}$ onto G, respectively.

$$\begin{array}{ccc} \overline{G} & \xrightarrow[\eta]{\eta_1} & \overline{Q} \\ & \searrow & \downarrow{\scriptstyle \eta_2} \\ & & G \end{array}$$

Figure 2

On the conjugacy classes of $\overline{G}$, $\overline{Q}$ and G

Let's consider $\overline{Q} = \frac{\overline{G}}{K} \cong P_1.G$, then G is identified with $\frac{\overline{Q}}{P_1}$ under the map η_2.
Moreover, under the map η_2, the pre-image of a conjugacy class $[P_1\overline{q}]$ in $\frac{\overline{Q}}{P_1}$ is a union $\bigcup_{i=1}^{\widehat{c(g)}}[\overline{q_i}]$ of say $\widehat{c(g)}$ conjugacy classes $[\overline{q_i}]$ in $\overline{Q}$. Note that each coset $P_1\overline{q}$ can be identified with a $g \in G$ such that $\overline{q}$ is a lifting for g.
Therefore, corresponding to a representative $P_1\overline{q} \in [P_1\overline{q}]$ (or a class representative $g \in G$) there is a set

$$\widehat{X(g)} = \{\overline{q_1} = \overline{q}, \overline{q_2}, \ldots, \overline{q_{\widehat{c(g)}}}\}$$

of representatives of conjugacy classes $[\overline{q_i}]$ of $\overline{Q}$.

Similarly, a pre-image of a class $[\overline{q_i}]$ of $\overline{Q}$, with $\overline{q_i} \in \widehat{X(g)}$, under the map η_1 will be a union $\bigcup_{j=1}^{c(\overline{q_i})}[\overline{g_{i_j}}]$ of $c(\overline{q_i})$ classes $[\overline{g_{i_j}}]$ of $\overline{G}$. Note that a $\overline{q_i} \in \widehat{X(g)}$ is identified with a coset $K\overline{g_i} \in \frac{\overline{G}}{K} \cong \overline{Q}$ where $\overline{g_i}$ is a lifting for $\overline{q_i}$ in $\overline{G}$. Hence a set

$$X(\overline{q_i}) = \{\overline{g_{i_1}} = \overline{g_i}, \overline{g_{i_2}}, \ldots, \overline{g_{i_{c(\overline{q_i})}}}\}$$

of representatives of conjugacy classes $[\overline{g_{i_j}}]$ is obtained from the coset $K\overline{g_i}$ (or equivalently a class representative $\overline{q_i} \in \overline{Q}$). Since $\eta = \eta_2 \circ \eta_1$ (see Figure 2), it follows that the pre-image for $g \in G$ under the map η is a set

$$\begin{aligned} X(g) &= \bigcup_{i=1}^{\widehat{c(g)}} X(\overline{q_i}) = X(\overline{q_1}) \bigcup X(\overline{q_2}) \bigcup \cdots \bigcup X(\overline{q_{\widehat{c(g)}}}) \\ &= \{\overline{g1_1}, \overline{g1_2}, \ldots, \overline{g1_{c(\overline{q_1})}}, \overline{g2_1}, \overline{g2_2}, \ldots, \overline{g2_{c(\overline{q_2})}}, \ldots, \overline{g_{\widehat{c(g)}_1}}, \overline{g_{\widehat{c(g)}_2}}, \ldots, \overline{g_{\widehat{c(g)}}}_{c(\overline{q_{\widehat{c(g)}}})}\} \end{aligned}$$

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

Since $\overline{Q}$ is a factor group of $\overline{G}$, the set $\text{Irr}(\overline{Q})$ can be lifted to $\overline{G}$ and the lifts are equivalent to characters $\chi_i \in \mathrm{Irr}(\overline{G})$ such that $K \leq \mathrm{Ker}(\chi_i)$. Moreover, the characters $\theta_i \in \mathrm{Irr}(P)$ of P such that $K \leq \mathrm{Ker}(\theta_i)$ are the lifts of $\widehat{\theta_i} \in \text{Irr}(P_1)$ to P. Hence the action of G on the lifts of $\text{Irr}(P_1)$ to P is identical as the action of G on $\text{Irr}(P_1)$, i.e. the number and lengths of the orbits of G on the lifts of $\text{Irr}(P_1)$ and $\text{Irr}(P_1)$ and their corresponding inertia factor groups H_i coincide. Suppose that G has s orbits on $\text{Irr}(P_1)$ with corresponding inertia factors H_i, $i = 1, 2, \ldots, s$. Then the blocks $M_i(g)$, $i = 1, 2, \ldots, s$, of the matrix $M(g)$ of $\overline{G}$ in Figure 3 will correspond to the matrix $\widehat{M(g)}$ of $\overline{Q}$.

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

The columns of $M(g)$ are indexed by the set $X(g)$. The columns (for the s $M_i(g)$ blocks) of $M(g)$ which correspond to the centralizers $C_{\overline{G}}(\overline{g_{i_1}})$ of class representatives $\overline{g_{i_1}} \in X(g)$, $i = 1, 2, \ldots, \widehat{c(g)}$ (see Figure 3) will just be the $\widehat{c(g)}$ columns of the matrix $\widehat{M(g)}$. Note that the elements $\overline{g_{i_1}}$ are the lifts of $\overline{q_i} \in \widehat{X(g)}$ to $\overline{G}$. Whereas the other columns of $M(g)$ (for the s $M_i(g)$ blocks) corresponding to the class representatives $\overline{g_{i_j}} \in X(g)$, $j = 2, \ldots, c(\overline{q_i})$, are duplicates of the columns of $M(g)$ associated with the class representatives $\overline{g_{i_1}} \in X(g)$, $i = 1, 2, \ldots, \widehat{c(g)}$, since the $\overline{g_{i_j}}$'s come from the coset $K\overline{g_i}$. Note if $\chi \in \mathrm{Irr}(\overline{G})$ is a lift for $\widehat{\chi} \in \mathrm{Irr}(\overline{Q})$ to $\overline{G}$, then $\chi(\overline{g_{i_j}}) = \widehat{\chi}(K\overline{g_i})$ where $K\overline{g_i}$ is identified with $\overline{q_i} \in \widehat{X(g)}$. For example, in Figure 3, the columns corresponding to the class representatives $\overline{g_{1_j}} \in X(q_1) \subseteq X(g)$, $j = 2, 3, \ldots, c(\overline{q_1})$ are the duplicates of the column which is indexed by $\overline{g_{1_1}} \in X(q_1)$, that is, $a^{1_1}_{(i,y_k)} = a^{1_{c(\overline{q_1})}}_{(i,y_k)}$ for $1 \leq i \leq s$, $1 \leq k \leq r$, since the $c(\overline{q_1})$ elements $\overline{g_{1_j}}$ come from the coset $K\overline{g_1}$.

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

$$
\begin{array}{c|cccc}
 & X(\overline{q_1}) & X(\overline{q_2}) & & X(\overline{q_{\widehat{c(g)}}}) \\
 & |C_{\overline{G}}(\overline{g_1}_1)| \ldots |C_{\overline{G}}(\overline{g_1}_{c(\overline{q_1})})| & |C_{\overline{G}}(\overline{g_2}_1)| \ldots |C_{\overline{G}}(\overline{g_2}_{c(\overline{q_2})})| & \cdots & |C_{\overline{G}}(\overline{g_{\widehat{c(g)}}}_1)| \ldots |C_{\overline{G}}(\overline{g_{\widehat{c(g)}}}_{c(\overline{q_{\widehat{c(g)}}})})| \\
\hline
|C_G(g)| & a^{1_1}_{(1,g)} \ldots a^{1_1}_{(1,g)} & a^{2_1}_{(1,g)} \ldots a^{2_1}_{(1,g)} & \cdots & a^{\widehat{c(g)}_1}_{(1,g)} \ldots a^{\widehat{c(g)}_1}_{(1,g)} \\
\hline
|C_{H_2}(y_1)| & a^{1_1}_{(2,y_1)} \ldots a^{1_1}_{(2,y_1)} & a^{2_1}_{(2,y_1)} \ldots a^{2_1}_{(2,y_1)} & \cdots & a^{\widehat{c(g)}_1}_{(2,y_1)} \ldots a^{\widehat{c(g)}_1}_{(2,y_1)} \\
|C_{H_2}(y_2)| & a^{1_1}_{(2,y_2)} \ldots a^{1_1}_{(2,y_2)} & a^{2_1}_{(2,y_2)} \ldots a^{2_1}_{(2,y_2)} & \cdots & a^{\widehat{c(g)}_1}_{(2,y_2)} \ldots a^{\widehat{c(g)}_1}_{(2,y_2)} \\
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
|C_{H_s}(y_1)| & a^{1_1}_{(s,y_1)} \ldots a^{1_1}_{(s,y_1)} & a^{2_1}_{(s,y_1)} \ldots a^{2_1}_{(s,y_1)} & \cdots & a^{\widehat{c(g)}_1}_{(s,y_1)} \ldots a^{\widehat{c(g)}_1}_{(s,y_1)} \\
|C_{H_s}(y_2)| & a^{1_1}_{(s,y_2)} \ldots a^{1_1}_{(s,y_2)} & a^{2_1}_{(s,y_2)} \ldots a^{2_1}_{(s,y_2)} & \cdots & a^{\widehat{c(g)}_1}_{(s,y_2)} \ldots a^{\widehat{c(g)}_1}_{(s,y_2)} \\
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
|C_{H_{s+1}}(y_1)| & a^{1_1}_{(s+1,y_1)} \ldots a^{1_{c(\overline{q_1})}}_{(s+1,y_1)} & a^{2_1}_{(s+1,y_1)} \ldots a^{2_{c(\overline{q_2})}}_{(s+1,y_1)} & \cdots & a^{\widehat{c(g)}_1}_{(s+1,y_1)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(s+1,y_1)} \\
|C_{H_{s+1}}(y_2)| & a^{1_1}_{(s+1,y_2)} \ldots a^{1_{c(\overline{q_1})}}_{(s+1,y_2)} & a^{2_1}_{(s+1,y_2)} \ldots a^{2_{c(\overline{q_2})}}_{(s+1,y_2)} & \cdots & a^{\widehat{c(g)}_1}_{(s+1,y_2)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(s+1,y_2)} \\
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
|C_{H_i}(y_1)| & a^{1_1}_{(i,y_1)} \ldots a^{1_{c(\overline{q_1})}}_{(i,y_1)} & a^{2_1}_{(i,y_1)} \ldots a^{2_{c(\overline{q_2})}}_{(i,y_1)} & \cdots & a^{\widehat{c(g)}_1}_{(i,y_1)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(i,y_1)} \\
|C_{H_i}(y_2)| & a^{1_1}_{(i,y_2)} \ldots a^{1_{c(\overline{q_1})}}_{(i,y_2)} & a^{2_1}_{(i,y_2)} \ldots a^{2_{c(\overline{q_2})}}_{(i,y_2)} & \cdots & a^{\widehat{c(g)}_1}_{(i,y_2)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(i,y_2)} \\
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
|C_{H_t}(y_1)| & a^{1_1}_{(t,y_1)} \ldots a^{1_{c(\overline{q_1})}}_{(t,y_1)} & a^{2_1}_{(t,y_1)} \ldots a^{2_{c(\overline{q_2})}}_{(t,y_1)} & \cdots & a^{\widehat{c(g)}_1}_{(t,y_1)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(t,y_1)} \\
|C_{H_t}(y_2)| & a^{1_1}_{(t,y_2)} \ldots a^{1_{c(\overline{q_1})}}_{(t,y_2)} & a^{2_1}_{(t,y_2)} \ldots a^{2_{c(\overline{q_2})}}_{(t,y_2)} & \cdots & a^{\widehat{c(g)}_1}_{(t,y_2)} \ldots a^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}_{(t,y_2)} \\
\vdots & \vdots \ldots \vdots & \vdots \ldots \vdots & \vdots & \vdots \ldots \vdots \\
\hline
 & m_{1_1} \ldots m_{1_{c(\overline{q_1})}} & m_{2_1} \ldots m_{1_{c(\overline{q_2})}} & \cdots & m_{\widehat{c(g)}_1} \ldots m_{1_{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}}}
\end{array}
$$

Figure 3: $M(g)$ of $\overline{G}$

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

Furthermore, suppose G has t orbits on $\text{Irr}(P)$ where s of the t orbits contain the lifts of $\text{Irr}(P_1)$ and the rest of the characters of $\text{Irr}(P)$ are in $t - s = b$ orbits with corresponding $H_{s+1}, H_{s+2}, \ldots, H_{s+b=t}$ inertia factor groups. The total number of α_i^{-1}-regular classes $[y_k]$ of the b inertia factors $H_{s+1}, H_{s+2}, \ldots, H_{s+b}$, that fuse into a class $[g]$ of G will be equal to $d = c(g) - \widehat{c(g)} = c(\overline{q_1}) + c(\overline{q_2}) + \cdots + c(\overline{q_{\widehat{c(g)}}}) - \widehat{c(g)}$. Therefore, to the s blocks of $M(g)$ containing $c(g)$ columns (as described above) and $\widehat{c(g)}$ rows, we will add d more rows which will be contained in a further b blocks corresponding to the inertia factors $H_{s+1}, H_{s+2}, \ldots, H_{s+b}$. The d rows will have the form,

$$\left[a_{(s+i,y_k)}^{1_1} \ldots a_{(s+i,y_k)}^{1_{c(\overline{q_1})}} a_{(s+i,y_k)}^{2_1} \ldots a_{(s+i,y_k)}^{2_{c(\overline{q_2})}} \ldots a_{(s+i,y_k)}^{\widehat{c(g)}_1} \ldots a_{(s+i,y_k)}^{\widehat{c(g)}_{c(\overline{q_{\widehat{c(g)}}})}} \right],$$

where $i = 1, 2, \ldots, b$ and $1 \leq k \leq r$.

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

Row and orthogonality relations for Fischer-Clifford matrices (see for example [1] or [16]) will be used to obtain the entries $a^{u_j}_{(i,y_k)}$, $u = 1, 2..., \widehat{c(g)}$, in the the blocks $M_i(g)$, $i = s+1, s+2, \ldots, s+b = t$, of the matrix $M(g)$ of $\overline{G}$. The final shape of a $M(g)$ of $\overline{G}$ is depicted in Figure 3. Note that $M(g)$ is a $(\widehat{c(g)} + d) \times (\widehat{c(g)} + d)$ matrix, which was obtained by adding d columns of sizes $\widehat{c(g)} \times 1$ and d rows of sizes $1 \times c(g)$ to the $\widehat{c(g)} \times \widehat{c(g)}$ matrix $\widehat{M(g)}$ of $\overline{Q}$. Hence we can formulate Theorem 0.1 below.

Theorem 0.1

For a class representative $g \in G$, the Fischer-Clifford matrix $\widehat{M(g)}$ of the quotient group $\overline{Q}$ is embedded (or contained) in the corresponding Fischer-Clifford matrix $M(g)$ of $\overline{G}$.

On the relationship between the Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

In a sense, we can say that the matrix $M(g)$ is a lift for $\widehat{M(g)}$ to $\overline{G}$. If there is no class fusion from the inertia factors $H_{s+1}, \ldots, H_{s+b}$ into $[g]$ then $M(g) = \widehat{M(g)}$ except possibly for changes in the class orders of $\overline{g_{i_j}} \in X(g)$. Having constructed the Fischer-Clifford matrices $M(g_i)$ of $\overline{G}$ from those matrices $\widehat{M(g_i)}$ of $\overline{Q}$ together with the fusion maps of the inertia factor groups $H_1, \ldots, H_s, H_{s+1}, \ldots, H_{s+b}$ into $H_1 = G$ and the sets $\mathrm{IrrProj}(H_i, \alpha_i^{-1})$, $i = 1, 2, \ldots, s, s+1, s+2, \ldots, s+b = t$, the set $\mathrm{Irr}(\overline{G})$ can be fully assembled.

Example: $N_{Th}(P) = 2^{4+5\cdot}A_8$ of a radical subgroup $P = 2^{4+5}$ in Th

- Let we consider a group of structure $\overline{G} = 2^{4+5\cdot}A_8$ which is the normalizer $N_{Th}(2^{4+5})$ of a radical 2-subgroup 2^{4+5} in the sporadic simple Thompson group Th (see [24]).
- $\overline{G}$ sits maximally in the 2nd largest maximal subgroup $D = 2^{5\cdot}GL_5(2)$ [6] of Th, called the Dempwolff group.
- $P = 2^{4+5}$ is a special 2-group of order 512, where the center $Z(P) = 2^4 \triangleleft \overline{G}$ since $Z(P)$ is a characteristic subgroup of P.
- Hence we can construct $\overline{Q} = \frac{\overline{G}}{Z(P)} \cong 2^{5\cdot}A_8$, where we let $P_1 = 2^5$
- The groups $\overline{G}$ and $\overline{Q}$ will be used to illustrate the "lifting of Fischer-Clifford matrices" technique.

Example: Actions of $\overline{G}$ and Q on $\text{Irr}(P)$ and $\text{Irr}(P_1)$

- $\overline{G}$ has five orbits on $\text{Irr}(P)$ with lengths 1, 1, 15, 15, and 120 with corresponding inertia factors $H_1 = H_2 = A_8$, $H_3 = H_4 = 2^3{:}GL_3(2)$ and $H_5 = 2^3{:}(7{:}3)$.
- Orbits of $\text{Irr}(P)$ having lengths 1, 1, 15 and 15 contain the lifts of the characters $\widehat{\chi}_i \in \text{Irr}(2^5)$ to P.
- Therefore, $\overline{Q}$ has four orbits of lengths 1, 1, 15, 15 on $\text{Irr}(2^5)$ where the corresponding inertia factors coincide with $H_1 = H_2 = A_8$ and $H_3 = H_4 = 2^3{:}GL_3(2)$.

Example:The Classes and Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

We will proceed to compute the Fischer-Clifford matrices $M(g_i)$ of $\overline{G}$ by adding an appropriate number of rows and columns to the matrices $\widehat{M(g_i)}$ of $\overline{Q}$ according to the number of classes $[y_k]$ of H_5 fusing into a class $[g]$ of A_8.

- Let consider the natural epimorphisms $\eta_2{:}\overline{Q} \to G$, $\eta_1{:}\overline{G} \to \overline{Q}$, $\eta = \eta_2 \circ \eta_1{:}\overline{G} \to G$ (see Figure 2) for $\overline{G} = 2^{4+5\cdot}A_8$ and $\overline{Q} = 2^{5\cdot}A_8$
- $\ker(\eta_1) = Z(P) = 2^4 = K$, $\ker(\eta_2) = 2^5 = P_1$ and $\ker(\eta) = 2^{4+5} = P$.
- The sets $\widehat{X(g)}$, $X(\overline{q_i})$ and $X(g) = \bigcup_{i=1}^{\widehat{c(g)}} X(\overline{q_i})$ with pre-images under η_2, η_1 and η, respectively

Example:The Classes and Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

As an example, let's consider the class $3B$ of G in Table 2 and we follow notation as discussed earlier.

- Under η_2 the set $\widehat{X(g)} = \{\overline{q_1} \in 3B, \overline{q_2} \in 6A, \overline{q_3} \in 6B, \overline{q_3} \in 6C\}$ of class representatives $\overline{q_i} \in \overline{Q}$ is obtained from a coset $P_1\overline{q}$, where $g \in 3B$ is identified with $P_1\overline{q}$ and we let $\overline{q_1} = \overline{q}$.
- The sets $X(\overline{q_1}) = \{\overline{g_1}_1 \in 3B, \overline{g_1}_2 \in 6B\}$, $X(\overline{q_2}) = \{\overline{g_2}_1 \in 6C\}$, $X(\overline{q_3}) = \{\overline{g_3}_1 \in 12A, \overline{g_3}_2 \in 12B\}$ and $X(\overline{q_4}) = \{\overline{g_4}_1 \in 6D\}$ of pre-images of $\overline{q_i} \in \widehat{X(g)}$ under η_1 are obtained.
- Since $\eta = \eta_2 \circ \eta_1$ it follows that the pre-images of a class representative $g \in 3B$ in G under η is the set $X(g) = \bigcup_{i=1}^{\widehat{c(g)}=4} X(\overline{q_i}) = \{\overline{g_1}_1, \overline{g_1}_2, \overline{g_2}_1, \overline{g_3}_1, \overline{g_3}_2, \overline{g_4}_1\}$.

Example:The Classes and Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

The Fischer-Clifford matrix $\widehat{M(3B)}$ which is associated with the class $3B$ of G is depicted as Figure 4. The columns are indexed by the orders $|C_{\overline{Q}}(\overline{q_i})|$ of the centralizers of class representatives $\overline{q_i} \in \widehat{X(g)}$, $g \in 3B$, and the rows are indexed by the orders of the centralizers $|C_{H_i}(y_1)|$of class representatives y_k, $k = 1$, of the inertia factors H_i, $i = 1, 2, 3, 4$, which fuse into the class $3B$ of G.

$$\widehat{M(3B)} = \begin{array}{c} \\ |C_{H_1}(y_1 \in 3B)| \\ |C_{H_2}(y_1 \in 3B)| \\ |C_{H_3}(y_1 \in 3A)| \\ |C_{H_4}(y_1 \in 3A)| \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\overline{Q}}(\overline{q_1})| & |C_{\overline{Q}}(\overline{q_2})| & |C_{\overline{Q}}(\overline{q_3})| & |C_{\overline{Q}}(\overline{q_4})| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \end{pmatrix} \end{array}$$

Figure 4: $\widehat{M(3B)}$ of $\overline{Q}$

Example:The Classes and Fischer-Clifford matrices of $\overline{G}$ and $\overline{Q}$

Now, the matrix $M(3B)$ will be constructed from $\widehat{M(3B)}$ following the "lifting technique" discussed above.

- The set $X(g) = \bigcup_{i=1}^{\widehat{c(g)}=4} X(\overline{q_i})$ contains the pre-images of $\overline{q_i} \in \widehat{X(g)}$ under η_1 and will index the columns for $M(3B)$.
- Since $\widehat{\chi}(\overline{q_i}) = \chi(\overline{g_{i_j}})$, for $\overline{g_{i_j}} \in X(\overline{q_i})$, $\widehat{\chi} \in \mathrm{Irr}(\overline{Q})$ and $\chi \in \mathrm{Irr}(\overline{G})$, the columns for $M(3B)$ corresponding to the blocks $M_i(3B)$, $i = 1, 2, 3, 4$ (see Figure 5), will just be duplicates of the columns of the matrix $\widehat{M(3B)}$.
- For example, the first column of $\widehat{M(3B)}$ labelled by $\overline{q_1} \in \widehat{X(g)}$ will be duplicated for the columns of $M(3B)$ which are labelled by the pre-images $\overline{g_{11}}, \overline{g_{12}}$ of $\overline{q_1}$ under η_1.
- Since both of the elements $\overline{q_2}$ and $\overline{q_4}$ have only one pre-image under η_1, the columns of $\widehat{M(3B)}$ labelled by $|C_{\overline{Q}}(\overline{q_2})|$ and $|C_{\overline{Q}}(\overline{q_4})|$ will only be repeated once for $M(3B)$, that is, the columns labelled by $|C_{\overline{G}}(\overline{g_{21}})|$ and $|C_{\overline{G}}(\overline{g_{41}})|$.

M(3B)=

	$X(\overline{q_1})$		$X(\overline{q_2})$	$X(\overline{q_3})$		$X(\overline{q_4})$
	$\lvert C_{\overline{G}}(\overline{g_{11}})\rvert$	$\lvert C_{\overline{G}}(\overline{g_{12}})\rvert$	$\lvert C_{\overline{G}}(\overline{g_{21}})\rvert$	$\lvert C_{\overline{G}}(\overline{g_{31}})\rvert$	$\lvert C_{\overline{G}}(\overline{g_{32}})\rvert$	$\lvert C_{\overline{G}}(\overline{g_{41}})\rvert$
$\lvert C_{H_1}(y_1 \in 3B)\rvert$	1	1	1	1	1	1
$\lvert C_{H_2}(y_1 \in 3B)\rvert$	1	1	-1	1	1	-1
$\lvert C_{H_3}(y_1 \in 3A)\rvert$	3	3	3	-1	-1	-1
$\lvert C_{H_4}(y_1 \in 3A)\rvert$	3	3	-3	-1	-1	1
$\lvert C_{H_5}(y_1 \in 3A)\rvert$	a	b	c	d	e	f
$\lvert C_{H_5}(y_2 \in 3B)\rvert$	g	h	i	j	k	l

Figure 5: $M(3B)$ of $\overline{G}$

- Furthermore, two classes $3A$ and $3B$ of the inertia factor H_5 fuse into the class $3B$ of G (see Table 1) and hence two more rows (rows 5 and 6 in Figure 5) will be added to complete the matrix $M(3B)$.
- The entries of rows five and six of $M(3B)$ are obtained by using the column and row orthogonality relations of Fischer-Clifford matrices and the desired matrix $M(3B)$ is found in Table 3.
- Note that the entries of the blocks $M_i(3B)$ of $M(3B)$ corresponding to the inertia factors H_i, $i = 1, 2, 3, 4$, are completely determined by the matrix $\widehat{M(3B)}$.
- Similarly, all other Fischer-Clifford matrices of $\overline{G}$ were computed and are listed in Table 3.

Example:The fusion maps of H_3 and H_5 into A_8

Table 1: **The fusion maps of H_3 and H_5 into A_8**

$[h]_{2^3:GL_3(2)}$	$\longrightarrow$	$[g]_{A_8}$	$[h]_{2^3:GL_3(2)}$	$\longrightarrow$	$[g]_{A_8}$
$1A$		$1A$	$4B$		$4A$
$2A$		$2A$	$4C$		$4B$
$2B$		$2A$	$6A$		$6B$
$2C$		$2B$	$7A$		$7B$
$3A$		$3B$	$7B$		$7A$
$4A$		$4A$			
$[h]_{2^3:(7:3)}$	$\longrightarrow$	$[g]_{A_8}$	$[h]_{2^3:(7:3)}$	$\longrightarrow$	$[g]_{A_8}$
$1A$		$1A$	$6A$		$6B$
$2A$		$2A$	$6B$		$6B$
$3A$		$3B$	$7A$		$7B$
$3B$		$3B$	$7B$		$7A$

The classes of $\overline{Q}$ and $\overline{G}$

Table 2: The conjugacy classes of $\overline{Q}$ and $\overline{G}$

$[g]_G$	k	f_j	$[q]_{\overline{Q}}$	$\lvert C_{\overline{Q}}(q)\rvert$	$\rightarrow [y]_{2.Co_1}$	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	$\rightarrow [y]_{2^5 \cdot GL_5(2)}$
$1A$	32	$f_1 = 1$	$1A$	645120	$1A$	$1A$	10321920	$1A$
						$2A$	688128	$2A$
		$f_2 = 1$	$2A$	645120	$2B$	$2B$	645120	$2A$
		$f_3 = 15$	$2B$	43008	$2C$	$2C$	43008	$2A$
		$f_4 = 15$	$2C$	43008	$2B$	$4A$	43008	$4A$
$2A$	16	$f_1 = 2$	$2D$	1536	$2B$	$4B$	3072	$4A$
						$2D$	3072	$2A$
		$f_2 = 6$	$2E$	512	$2D$	$4C$	512	$4B$
		$f_3 = 8$	$4A$	384	$4G$	$8A$	384	$8B$
$2B$	8	$f_1 = 2$	$4B$	384	$4E$	$4D$	384	$4B$
		$f_2 = 6$	$4C$	128	$4D$	$4E$	128	$4B$
$3A$	2	$f_1 = 1$	$6A$	360	$6I$	$6A$	360	$6B$
		$f_2 = 1$	$3A$	360	$3C$	$3A$	360	$3B$
$3B$	8	$f_1 = 1$	$3B$	144	$3B$	$3B$	576	$3A$
						$6B$	192	$6A$
		$f_2 = 1$	$6B$	144	$6K$	$6C$	144	$6A$
		$f_3 = 3$	$6C$	48	$6L$	$12A$	96	$12A$
						$12B$	96	$12A$
		$f_4 = 3$	$6D$	48	$6K$	$6D$	48	$6A$
$4A$	8	$f_1 = 2$	$4D$	64	$4E$	$4F$	64	$4A$
		$f_2 = 2$	$4E$	64	$4G$	$8B$	64	$8A$
		$f_3 = 4$	$4F$	32	$4H$	$8C$	32	$8A$
$4B$	4	$f_1 = 2$	$8A$	16	$8G$	$8D$	16	$8B$
		$f_2 = 2$	$8B$	16	$8F$	$8E$	16	$8B$
$5A$	2	$f_1 = 1$	$5A$	30	$5C$	$5A$	30	$5A$
		$f_2 = 1$	$10A$	30	$10H$	$10A$	30	$10A$
$6A$	2	$f_1 = 2$	$12A$	12	$12L$	$12C$	12	$12C$
$6B$	4	$f_1 = 1$	$6E$	24	$6K$	$12D$	48	$12C$
						$6E$	48	$6C$
		$f_2 = 1$	$6F$	24	$6K$	$12E$	48	$12C$
						$6F$	48	$6C$
		$f_3 = 1$	$12B$	24	$12P$	$24A$	24	$24A$
		$f_4 = 1$	$12C$	24	$12P$	$24B$	24	$24A$
$7A$	4	$f_1 = 1$	$7A$	28	$7B$	$7A$	56	$7A$
						$14A$	56	$14A$
		$f_2 = 1$	$14A$	28	$14C$	$28A$	28	$28A$
		$f_3 = 1$	$14B$	28	$14C$	$14B$	28	$14A$
		$f_4 = 1$	$14C$	28	$14D$	$14C$	28	$14A$
$7B$	4	$f_1 = 1$	$7B$	28	$7B$	$7B$	56	$7A$
						$14D$	56	$14D$
		$f_2 = 1$	$14D$	28	$14C$	$14E$	28	$14A$
		$f_3 = 1$	$14E$	28	$14D$	$14F$	28	$14A$
		$f_4 = 1$	$14F$	28	$14C$	$28B$	28	$28A$
$15A$	2	$f_1 = 1$	$15A$	30	$15E$	$15A$	30	$15A$
		$f_2 = 1$	$30A$	30	$30J$	$30A$	30	$30A$
$15B$	2	$f_1 = 1$	$15B$	30	$15E$	$15B$	30	$15B$
		$f_2 = 1$	$30B$	30	$30J$	$30B$	30	$30B$

The Fischer-Clifford Matrices of and $\overline{Q}$ and $\overline{G}$

Table 3: The Fischer-Clifford Matrices of $\overline{Q}$ and $\overline{G}$

$\widehat{M(1A)}$	$\widehat{M(2A)}$	$M(1A)$	$M(2A)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 15 & -1 & -1 \\ 15 & -15 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 15 & 15 & 15 & -1 & -1 \\ 15 & 15 & -15 & 1 & -1 \\ 240 & -16 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 6 & 6 & -2 & 0 \\ 8 & -8 & 0 & 0 \end{pmatrix}$
$\widehat{M(2B)}$	$\widehat{M(3A)}$	$M(2B)$	$M(3A)$
$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$\widehat{M(3B)}$	$\widehat{M(4A)}$	$M(3B)$	$M(4A)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -3 & -1 & -1 & 1 \\ 6 & -2 & 0 & 2 & -2 & 0 \\ 6 & -2 & 0 & -2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$
$\widehat{M(4B)}$	$\widehat{M(5A)}$	$M(4B)$	$M(5A)$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$\widehat{M(6A)}$	$\widehat{M(6B)}$	$M(6B)$	$M(6A)$
$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \end{pmatrix}$
$\widehat{M(7A)}$	$\widehat{M(7B)}$	$M(7A)$	$M(7B)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$
$\widehat{M(15A)}$	$\widehat{M(15B)}$	$M(15A)$	$M(15B)$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Table 4: The character tables of $\overline{Q}$ and $\overline{G}$

$[g]_{A_8}$	1A					2A				2B		3A	
$[q]_{\overline{Q}}$	1A		2A	2B	2C	2D		2E	4A	4B	4C	6A	3A
$[\overline{g}]_{\overline{G}}$	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E	6A	3A
p=2	1A	1A	1A	1A	2A	2A	1A	2A	4A	2B	2C	3A	3A
p=3	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E	2B	1A
p=5	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E	6A	3A
p=7	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E	6A	3A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	7	7	7	-1	-1	-1	-1	3	3	4	4
χ_3	14	14	14	14	14	6	6	6	6	2	2	-1	-1
χ_4	20	20	20	20	20	4	4	4	4	4	4	5	5
χ_5	21	21	21	21	21	-3	-3	-3	-3	1	1	6	6
χ_6	21	21	21	21	21	-3	-3	-3	-3	1	1	-3	-3
χ_7	21	21	21	21	21	-3	-3	-3	-3	1	1	-3	-3
χ_8	28	28	28	28	28	-4	-4	-4	-4	4	4	1	1
χ_9	35	35	35	35	35	3	3	3	3	-5	-5	5	5
χ_{10}	45	45	45	45	45	-3	-3	-3	-3	-3	-3	0	0
χ_{11}	45	45	45	45	45	-3	-3	-3	-3	-3	-3	0	0
χ_{12}	56	56	56	56	56	8	8	8	8	0	0	-4	-4
χ_{13}	64	64	64	64	64	0	0	0	0	0	0	4	4
χ_{14}	70	70	70	70	70	-2	-2	-2	-2	2	2	-5	-5
χ_{15}	8	8	-8	-8	8	0	0	0	0	0	0	4	-4
χ_{16}	24	24	-24	-24	24	0	0	0	0	0	0	6	-6
χ_{17}	24	24	-24	-24	24	0	0	0	0	0	0	6	-6
χ_{18}	48	48	-48	-48	48	0	0	0	0	0	0	-6	6
χ_{19}	56	56	-56	-56	56	0	0	0	0	0	0	4	-4
χ_{20}	56	56	-56	-56	56	0	0	0	0	0	0	4	-4
χ_{21}	56	56	-56	-56	56	0	0	0	0	0	0	-2	2
χ_{22}	56	56	-56	-56	56	0	0	0	0	0	0	-2	2
χ_{23}	64	64	-64	-64	64	0	0	0	0	0	0	-4	4
χ_{24}	15	15	15	-1	-1	7	7	-1	-1	3	-1	0	0
χ_{25}	45	45	45	-3	-3	-3	-3	5	-3	-3	1	0	0
χ_{26}	45	45	45	-3	-3	-3	-3	5	-3	-3	1	0	0
χ_{27}	90	90	90	-6	-6	18	18	2	-6	6	-2	0	0
χ_{28}	105	105	105	-7	-7	-7	-7	1	1	9	-3	0	0
χ_{29}	105	105	105	-7	-7	1	1	9	-7	-3	1	0	0
χ_{30}	105	105	105	-7	-7	17	17	-7	1	-3	1	0	0
χ_{31}	120	120	120	-8	-8	8	8	8	-8	0	0	0	0
χ_{32}	210	210	210	-14	-14	10	10	-6	2	6	-2	0	0
χ_{33}	315	315	315	-21	-21	-21	-21	3	3	3	-1	0	0
χ_{34}	315	315	315	-21	-21	3	3	-5	3	-9	3	0	0
χ_{35}	120	120	-120	8	-8	0	0	0	0	0	0	0	0
χ_{36}	120	120	-120	8	-8	0	0	0	0	0	0	0	0
χ_{37}	120	120	-120	8	-8	0	0	0	0	0	0	0	0
χ_{38}	360	360	-360	24	-24	0	0	0	0	0	0	0	0
χ_{39}	360	360	-360	24	-24	0	0	0	0	0	0	0	0
χ_{40}	240	-16	0	0	0	8	-8	0	0	0	0	0	0
χ_{41}	240	-16	0	0	0	8	-8	0	0	0	0	0	0
χ_{42}	240	-16	0	0	0	8	-8	0	0	0	0	0	0
χ_{43}	720	-48	0	0	0	24	-24	0	0	0	0	0	0
χ_{44}	720	-48	0	0	0	24	-24	0	0	0	0	0	0
χ_{45}	1680	-112	0	0	0	-8	8	0	0	0	0	0	0
χ_{46}	1680	-112	0	0	0	-8	8	0	0	0	0	0	0
χ_{47}	1680	-112	0	0	0	-8	8	0	0	0	0	0	0

The Character Table of $\overline{G}$

Table 4 (continued)

$[g]_{A_8}$	3B						4A			4B		5A		6A
$[q]_{\overline{Q}}$	3B		6B	6C		6D	4D	4E	4F	8A	8B	5A	10A	12A
$[\overline{g}]_{\overline{G}}$	3B	6B	6C	12A	12B	6D	8B	4F	8C	8D	8E	5A	10A	12C
$p=2$	3B	3B	3B	6B	6B	3B	4B	2D	4C	4D	4E	5A	5A	6A
$p=3$	1A	2A	2B	4A	4A	2C	8B	4F	8C	8D	8E	5A	10A	4D
$p=5$	3B	6B	6C	12B	12A	6D	8B	4F	8C	8D	8E	1A	2B	12C
$p=7$	3B	6B	6C	12A	12B	6D	8B	4F	8C	8D	8E	5A	10A	12C
χ_{1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{2}	1	1	1	1	1	1	-1	-1	-1	1	1	2	2	0
χ_{3}	2	2	2	2	2	2	2	2	2	0	0	-1	-1	-1
χ_{4}	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	1
χ_{5}	0	0	0	0	0	0	1	1	1	-1	-1	1	1	-2
χ_{6}	0	0	0	0	0	0	1	1	1	-1	-1	1	1	1
χ_{7}	0	0	0	0	0	0	1	1	1	-1	-1	1	1	1
χ_{8}	1	1	1	1	1	1	0	0	0	0	0	-2	-2	1
χ_{9}	2	2	2	2	2	2	-1	-1	-1	-1	-1	0	0	1
χ_{10}	0	0	0	0	0	0	1	1	1	1	1	0	0	0
χ_{11}	0	0	0	0	0	0	1	1	1	1	1	0	0	0
χ_{12}	-1	-1	-1	-1	-1	-1	0	0	0	0	0	1	1	0
χ_{13}	-2	-2	-2	-2	-2	-2	0	0	0	0	0	-1	-1	0
χ_{14}	1	1	1	1	1	1	-2	-2	-2	0	0	0	0	-1
χ_{15}	2	2	-2	2	2	-2	0	0	0	0	0	-2	2	0
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
χ_{17}	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
χ_{18}	0	0	0	0	0	0	0	0	0	0	0	-2	2	0
χ_{19}	-1	-1	1	-1	-1	1	0	0	0	0	0	1	-1	0
χ_{20}	-1	-1	1	-1	-1	1	0	0	0	0	0	1	-1	0
χ_{21}	2	2	-2	2	2	-2	0	0	0	0	0	1	-1	0
χ_{22}	2	2	-2	2	2	-2	0	0	0	0	0	1	-1	0
χ_{23}	-2	-2	2	-2	-2	2	0	0	0	0	0	-1	1	0
χ_{24}	3	3	3	-1	-1	-1	3	-1	-1	1	-1	0	0	0
χ_{25}	0	0	0	0	0	0	1	-3	1	1	-1	0	0	0
χ_{26}	0	0	0	0	0	0	1	-3	1	1	-1	0	0	0
χ_{27}	0	0	0	0	0	0	2	2	-2	0	0	0	0	0
χ_{28}	3	3	3	-1	-1	-1	-3	1	1	1	-1	0	0	0
χ_{29}	3	3	3	-1	-1	-1	-3	1	1	-1	1	0	0	0
χ_{30}	3	3	3	-1	-1	-1	1	-3	1	-1	1	0	0	0
χ_{31}	-3	-3	-3	1	1	1	0	0	0	0	0	0	0	0
χ_{32}	-3	-3	-3	1	1	1	-2	-2	2	0	0	0	0	0
χ_{33}	0	0	0	0	0	0	3	-1	-1	-1	1	0	0	0
χ_{34}	0	0	0	0	0	0	-1	3	-1	1	-1	0	0	0
χ_{35}	6	6	-6	-2	-2	2	0	0	0	0	0	0	0	0
χ_{36}	-3	-3	3	1	1	-1	0	0	0	0	0	0	0	0
χ_{37}	-3	-3	3	1	1	-1	0	0	0	0	0	0	0	0
χ_{38}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	12	-4	0	0	0	0	0	0	0	0	0	0	0	0
χ_{41}	-6	2	0	A	-A	0	0	0	0	0	0	0	0	0
χ_{42}	-6	2	0	-A	A	0	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	12	-4	0	0	0	0	0	0	0	0	0	0	0	0
χ_{46}	-6	2	0	A	-A	0	0	0	0	0	0	0	0	0
χ_{47}	-6	2	0	-A	A	0	0	0	0	0	0	0	0	0

$A = -2\sqrt{3}i$

Examples of groups suitable for "Lifting" technique

The following p-local maximal subgroups of the largest sporadic Monster simple group $\mathbb{M}$ and the Conway group Co_3 are suitable candidates to apply the "lifting of Fischer-Clifford matrices":

- $7_+^{1+4}{:}(3 \times 2S_7)$
- $2^{9+16} \cdot Sp_8(2)$
- $2^{5+10+20} \cdot (S_3 \times L_5(2))$
- $2^{3+6+18} \cdot (L_3(2) \times 3S_6)$
- $2^{2+12}{:}(A_8 \times S_3)$

Acknowledgments

The NRF grant for rated researchers (grant no. 132257) supported the research work done in this paper. I am most grateful to my Lord Jesus Christ.

The Bibliography

[1] F. Ali and J. Moori, *The Fischer-Clifford matrices of a maximal subgroup of* Fi'_{24}, Representation Theory, **7** (2003), 300-321.

[2] A.Basheer and J. Moori, *Fischer matrices of Dempwolff group* $2^{5\cdot}GL(5,2)$, Int. J. Group Theory, Vol. **1** No.4 (2012), 43-63.

[3] A. B. M. Basheer and J. Moori, *On a Maximal Subgroup of the Affine General Linear Group of* $GL(6,2)$, Advances in Group Theory and Applications, **11** (2021), 1-30.

[4] W. Bosma and J.J. Canon, *Handbook of Magma Functions*, Department of Mathematics, University of Sydney, November 1994.

[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.

[6] U. Dempwolff, *On extensions of elementary abelian groups of order* 2^5 *by* $GL(5,2)$, Rend. Sem. Mat. Univ. Padova, **48**(1972), 359 - 364.

[7] B. Fischer, *Clifford-matrices*, Progr. Math. **95**, Michler G.O. and Ringel C.(eds), Birkhauser, Basel (1991), 1 - 16.

[8] D. Gorenstein, *Finite Groups*, Harper and Row Publishers, New York, 1968.

[9] R.J. Haggarty and J.F. Humphreys, *Projective characters of finite groups*, Proc. London Math. Soc. (3)**36** (1975), 176 - 192.

[10] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, San Diego, 1976.

[11] G. Karpilovsky, *Group Representations: Introduction to Group Representations and Characters*, Vol 1 Part B, North - Holland Mathematics Studies 175, Amsterdam, 1992.

[12] G. Karpilovsky, *Projective representations of finite groups*, Marcel Dekker, New York and Basel, 1985.

[13] K. Lux and H. Pahlings, *Representations of Groups: A Computational Approach*, Cambridge University Press, Cambridge, 2010

[14] J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Maths Soc, Vol **2** (1981), 61 - 67.

[15] Z. Mpono, *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, 1998.

[16] J. Moori and Z.E. Mpono, *The Fischer-Clifford matrices of the group* $2^6{:}SP_6(2)$, Quaestiones Mathematicae, **22** (1999), 257-298.

[17] H. Pahlings, *The character table of* $2^{1+22\cdot}Co_2$, J. Algebra **315** (2007), no. 1,301-325

[18] A.L. Prins, *On the projective character tables of the maximal subgroups of* M_{11}*,* M_{12} *and* $Aut(M_{12})$, Advances in Group Theory and Applications, in press.

[19] A.L. Prins, The character table of an involution centralizer in the Dempwolff group $2^{5\cdot}GL_5(2)$, *Quaestiones Mathematicae* **39** (2016), 561-576.

[20] R.L. Fray, R.L. Monaledi and A.L. Prins, *Fischer-Clifford matrices of a group* $2^8{:}(U_4(2){:}2)$ *as a subgroup of* $O^+_{10}(2)$, Afr. Mat., **27** (2016), 1295-1310.

[21] The GAP Group, *GAP --Groups, Algorithms, and Programming*, Version 4.6.3; 2013. (http://www.gap-system.org).

[22] N.S. Whitley, *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Natal, 1994.

[23] R.A. Wilson, P. Walsh, J. Tripp, I. Suleiman, S. Rogers, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbot, *ATLAS of Finite Group Representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

[24] S. Yoshiara, *The radical 2-subgroups of the sporadic simple groups* J_4*,* Co_2 *and Th*, J. Algebra, 233 (2000), 309-341