A comparative study of a multidomain spectral quasilinearization and block hybrid methods for solving evolution parabolic equation

Mapule Pheko

School of Mathematical and Statistical Sciences
North-West University, Mafikeng Campus
REPUBLIC OF SOUTH AFRICA

Letlhogonolo D. Moleleki, Sicelo P. Goqo

65th Annual congress of South African Mathematical Society
06-08 December 2022

NORTH-WEST UNIVERSITY
NOORDWES-UNIVERSITEIT
YUNIBESITI YA BOKONE-BOPHIRIMA

Outline

Introduction

- An evolution equation is a PDE that explains the time evolution of a physical system starting from given initial data.
- Evolution equations appear in a range of applied and engineering sciences.
- The improvement of numerical and analytical techniques for solving complex, significantly nonlinear evolution PDEs continues to attract the attention of scientists who aim to improve a better understanding of such appealing nonlinear problems.
- Many numerical methods have been developed by academics and computational efficiency can still be improved.
- And some numerical techniques for approximating NLPDEs have been proposed.
- These include, but not limited to, the spectral method, the block-hybrid method and the finite difference method.
- To improve computing capacity, it is necessary to develop new schemes and improve existing numerical methods.

- The multidomain bivariate spectral quasilinearization method (MD-BSQLM) is an improvement on the BSQLM.
- The multidomain method is only used in the time domain.
- MD-BSQLM can be used on equations with $t > 1$ and the domain is divided into nonoverlapping sub-intervals.
- The block hybrid method, is a self-starting scheme and is performed as a block, providing more accurate results.
- The use of hybrid (off-step) points in block techniques provides numerous benefits, in the same way as the capability to adjust step size, use information off-step points, and minimize the zero stability barrier condition.
- According to the researchers, both the methods show a very high accuracy when compared to the exact solutions, hence they are very competitive with existing methods.

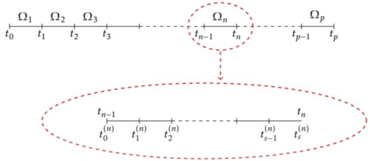
Aims and Objectives

- The aim of this study is to impliment the MD-BSQLM and Block-Hybrid method to solve nonlinear evolution problems.
- The main objective is to analyze the results obtained by the BHM and compare them with the MD-BSQLM results when solving nonlinear evolution problems.
- The accuracy, CPU time, and efficiency of all the methods will be investigated.

MD-BSQLM

The multidomain bivariate spectral quasilinearization method is a technique that uses the multidomain approach in the time domain only, bivariate Lagrange interpolation polynomial and the ideas of Newton-Raphson method to linearize the nonlinear ordinary and partial differential equations.

We by break down the interval $\Omega = [0, T]$ into intervals that are not overlapping $\Omega_k = [t_{k-1}, t_k]$ where $k = 1, 2, 3, ..., p$, whereas $t_0 = 0$ with $t_p = T$. The foremost notion behind this technique is to determine the result of the differential equation singly on every sub-interval, in succession, starting with the initial conditiont. A figure representation of the multi-domain grid points is shown below



To illustrate the development of the MD-BSQLM iteration scheme we consider a general the following nonlinear partial differential eqaution

$$\frac{\partial u}{\partial t} = G\Big(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\Big),\ (x,t) \in [a,b] \times [0,T], \tag{1}$$

subject to initial and boundary conditions

$$u(x,0) = u_0,\ u(a,t) = g_a(t),\ u(b,t) = g_b(t), \tag{2}$$

where G is a non-linear operator that hold any of u's spatial derivatives.

When rephrasing Eq. (1) in a linear structure before expressing it, it is appropriate to divide G into linear and nonlinear operators,

$$T[u, u', \ldots, u^{(n)}] + K[u, u', \ldots, u^{(n)}] - \dot{u} = 0, \tag{3}$$

where T is a linear operator, and K is a non-linear operator. We linearize K using Taylor series expansion

$$K[u, u', \ldots, u^{(n)}] \approx K[u, u', \ldots, u^{(n)}] + \sum_{p=0}^{n} \frac{\partial K}{\partial u^{(p)}}(u_{s+1}^{(p)} - u_s^{(p)}), \tag{4}$$

where s and $s+1$ represent the previous and current iterations, respectively. Eq. (4) can be written as

$$K[u,u',\dots,u^{(n)}] \approx K[u,u',\dots,u^{(n)}] + \sum_{p=0}^{n} a_{p,s}[u,u',\dots,u^{(n)}]u_{s+1}^{(p)} - a_{p,s}[u,u',\dots,u^{(n)}]u_s^{(p)}, \tag{5}$$

where

$$a_{p,s}[u,u',\dots,u^{(n)}] = \frac{\partial K}{\partial u^{(p)}}[u,u',\dots,u^{(n)}]. \tag{6}$$

Substituting equation (5) into (3), we get

$$T[u_{s+1},u'_{s+1},\dots,u_{s+1}^{(n)}] + \sum_{p=0}^{n} a_{p,s}u_{s+1}^{(p)} - \dot{u}_{s+1} = R_s[u_s,u'_s,\dots,u_s^{(n)}], \tag{7}$$

where

$$R_s[u_s,u'_s,\dots,u_s^{(n)}] = \sum_{p=0}^{n} a_{p,s}u_s^{(p)} - G[u_s,u'_s,\dots,u_s^{(n)}].$$

Thus, in each $k = 1,2,\dots p$ sub-interval, we must solve

$$a_{0,s}(x,t)u_{s+1}^{(n)(k)}(x,t) + a_{1,s}(x,t)u_{s+1}^{(n-1)(k)}(x,t) + \cdots + a_{2,s}(x,t)u_{s+1}^{(k)}(x,t) - \dot{u}_{s+1}^{(k)} = R_s^{(k)}(x,t), \quad x \in [a,b], t \in [t_{k-1},t_k], \tag{8}$$

Block-Hybrid Method

The development of block hybrid linear multi-step methods with off-step points for the result of nonlinear differential equation systems is considered.

To introduce the block method algorithms, we consider one-step methods for solving first IVPs and parabolic differential equations. The first order IVPs are solved over an interval with $0 \leq x \leq T$ which is partitioned as

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = T$$

with the step length defined as $h = x_{n+1} - x_n$ for $n = 0, 1, \ldots, N-1$. The differential equation below (10) is solved in the N non-overlapping blocks, $[t_n, t_{n+1}]$, using the known initial condition $y(t_n)$ for $n = 0, 1, \ldots, N-1$. The list of points used in the solution process in each block $[t_n, t_{n+1}]$ is

$$t_n, t_{n+p_1}, t_{n+p_2}, \ldots, t_{n+p_{M-1}}, t_{n+p_M}$$

where $t_{n+p_M} = t_{n+1}$, with $p_M = 1$. A pictorial representation of the intra-step points is show below

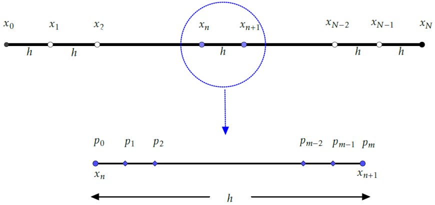


Figure: Distribution of intra-step points in the $[x_n, x_{n+1}]$ block

We approximate the exact result of general second-order PDE in the nature of,

$$\dot{y} = f(x, t, y, y', y''), \tag{10}$$

subject to initial and boundary conditions

$$y(x, 0) = y_0(x), \;\; y(a, t) = y_a(t) \;\; y(b, t) = y_b(t), \tag{11}$$

where $y_0(x)$, $y_a(t)$ and $y_b(t)$ are known functions. The PDE Eq. (10) is solved using the BHM in time t and the spectral collocation method in space x.

Consider linear PDE, which is expressed in general for second order as follows,

$$f(x,t,y,y',y'') = q_2(x,t)y''(x,t) + q_1(x,t)y'(x,t) + q_0(x,t)y(x,t)q + g(x,t),$$

where $g(x,t)$ and $q_k(x,t)$ for $k = 0,1,2$ are known fuctions. NLPDEs are linearised before the spectral method is applied. Applying the spectral method with $N_x + 1$ collocation points on (10) gives,

$$\dot{\mathbf{y}} = f(t,\mathbf{y}),\ \ \mathbf{y}(0) = \mathbf{y}_0, \tag{12}$$

and

$$f(t,\mathbf{y}) = [\mathbf{Q}_2(t)\mathbf{D}^2 + \mathbf{Q}_1(t)\mathbf{D} + \mathbf{Q}_0(t)]\mathbf{y} + \mathbf{G}(\mathbf{t}). \tag{13}$$

The continuous method works by approximating the analytical solution $y(t)$ of the non-linear differential equation of Eq. (10) by

$$y(t) \approx Y(t) = \sum_{i=0}^{M+1} c_{n,i}(t-t_n)^i, \tag{14}$$

where $t \in [a,b]$, $c_{n,i}$ are unknown coefficients to be determined and M is the number of collocation points. The continuous approximation is established by enforcing the following conditions:

$$\begin{aligned} \dot{Y}_{n+p_i} &= f(t_{n+p_i}, y_{n+p_i}),\ \ i = 0,1,2,\dots M, \\ Y(t_n) &= c_{n,0} = y_n,\ \ n = 0,1,\dots N-1, \end{aligned} \tag{15}$$

Working out the equations that emerge in Eq. (15), gives $c_{n,0} = y_n$ and $c_{n,1} = f_n$ for all values of M. The coefficients $c_{n,k}$ for $k \geq 2$ are

when $M = 2$,

$$c_{n,2} = -\frac{3f_n + f_{n+1} - 4f_{n+\frac{1}{2}}}{2h}, \quad c_{n,3} = \frac{2(f_n + f_{n+1} - 2f_{n+\frac{1}{2}})}{3h^2}.$$

The BHM equations are then obtained by subtituting the solutions for $c_{n,k}$ in the continuous approximation $Y(t)$ and evaluating the result at the collocation points t_{n+p_i} for $i = 1, 2 \ldots, M$. That is,

$$y_{n+p_i} = Y(t_{n+p_i}), \quad i = 1, 2, \ldots, M,$$

for equally spaced nodes $(0, \frac{1}{2}, 1)$ with $M = 2$, we obtain

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{24}h(5f_n - f_{n+1} + 8f_{n+\frac{1}{2}}),$$
$$y_{n+1} = y_n + \frac{1}{6}h(f_n + f_{n+1} + 4f_{n+\frac{1}{2}}).$$

Applying the BHM on the first order Eq. (12) gives

$$\mathbf{y}_{n+p_i} = \mathbf{y}_n + h\sum_{j=1}^{M}[\alpha_{i,j}\mathbf{f}_{n+p_j} + \beta_{i,j}\mathbf{f}_n], \; i = 1, 2, \ldots, M. \tag{16}$$

Where $\mathbf{f}_{n+p_j} = f(t_{n+p_j}, \mathbf{y}_{n+p_j})$ and $\mathbf{f}_n = f(t_n, \mathbf{y}_n)$. For general linear PDEs, spectral method decomposed scheme Eq. (16) can be expressed as

$$\mathbf{y}_{n+p_i} = \mathbf{y}_n + h\sum_{j=1}^{M}[\alpha_{i,j}(\mathbf{\Phi}_{n+p_j}\mathbf{y}_{n+p_j} + \mathbf{\Psi}_{n+p_j}) + \beta_{i,j}\mathbf{f}_n], \; i = 1, 2, \ldots, M, \tag{17}$$

where

$$\mathbf{\Phi}_{n+p_i} = \mathbf{Q}_{v,n+p_i}\mathbf{D}^{v} + \mathbf{Q}_{v-1,n+p_i}\mathbf{D}^{v-1} + \cdots + \mathbf{Q}_{1,n+p_i}\mathbf{D} + \mathbf{Q}_{0,n+p_i},$$

and

$$\mathbf{A}_{i,j} = \begin{cases} -h\alpha_{i,j}\mathbf{\Phi}_{n+p_j} & i \neq j \\ \mathbf{I} - h\alpha_{i,i}\mathbf{\Phi}_{n+p_i} & i = j \end{cases} \qquad B_{i,j} = h\beta_{i,j}\mathbf{I}, \;\; C_{i,j} = h\alpha_{i,j}\mathbf{I}.$$

A compact representation of Eq. (17) is

$$A_M Y_{n+M} = Y_n + B_m F_n + C_M \mathbf{\Psi}_{n+M} f_{n+M}, \tag{18}$$

Numerical Solution

Example : We consider the generalized Burgers-Fisher equation [1]

$$\frac{\partial u}{\partial t} + \alpha u^n \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u, \quad (x,t) \in [0,1] \times [0,10], \tag{19}$$

subject to initial condition

$$u(x,0) = \left(\frac{1}{2} + \frac{1}{2} \tanh(a_1 x) \right)^{\frac{1}{n}}, \tag{20}$$

and exact solution

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2} \tanh\left[a_1(x - a_2 t)\right] \right)^{\frac{1}{n}}, \tag{21}$$

where $a_1 = \frac{-\alpha n}{2(1+n)}$ and $a_2 = \frac{\alpha}{1+n} + \frac{\beta(1+n)}{\alpha}$. We chose these parameters to be $\alpha = \beta = n = 1$ for illustration purposes.

The appropriate nonlinear operator G for this example using MD-BSQLM is chosen as

$$G(u, u', u'') = u'' - \alpha u u' - \beta u^2 + \beta u. \tag{22}$$

For the purposes of this example using BHM, the suitable nonlinear operator G is taken as

$$\dot{u} \equiv G(u, u', u'') = u'' - \alpha u u' - \beta u^2 + \beta u, \tag{23}$$

thus

$$\left.\begin{array}{l} \mathbf{\Phi}_{n+p_j} = \mathbf{D}^2 + \text{diag}(-\nu u)\mathbf{D} + \text{diag}(\alpha u' + \beta - 2\beta u)\mathbf{I}, \text{ and} \\ \mathbf{\Psi}_{n+p_j} = \alpha u u' - \beta u^2. \end{array}\right\} \tag{24}$$

We use Taylor series expansion to linearize the nonlinear operators G.

Results and Discussion

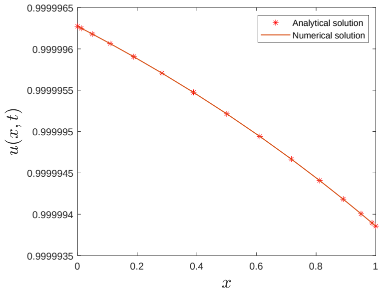


Figure: Exact vs Numerical of Burgers-Fisher equation for MD-BSQLM

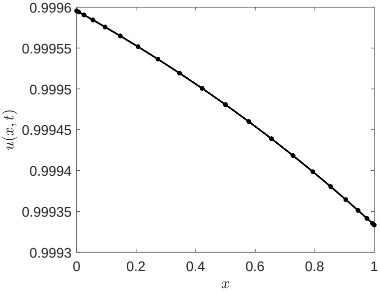


Figure: Exact vs Numerical solution of Burgers-Fisher equation for BHM

$t \backslash N_x$	14	16	18	20
0.50	1.972e-13	7.550e-15	7.849e-13	5.360e-13
2.50	4.663e-14	1.295e-13	1.674e-13	1.132e-14
5.00	2.620e-13	2.398e-14	8.116e-14	3.847e-13
7.50	1.110e-14	2.901e-13	1.927e-12	3.058e-12
10.00	2.224e-13	5.298e-13	1.096e-12	6.750e-14
CPU time(sec)	0.061981	0.072012	0.098814	0.103631

Table: Maximum errors for Burgers-Fisher equation using MD-BSQLM

t	$M = 3$	$M = 4$	$M = 5$	$M = 6$
0.00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
2.50	7.4933e-09	3.2794e-10	5.0350e-12	1.0514e-13
5.00	8.3324e-10	3.1085e-11	4.4287e-13	6.2395e-14
7.50	2.2348e-10	1.7964e-11	2.2182e-13	7.0055e-14
10.00	1.5796e-10	1.2352e-11	3.2951e-13	9.2371e-14
CPU time(sec)	0.190243	0.186671	0.320958	0.385719

Table: Maximum errors for Burgers-Fisher equation using BHM

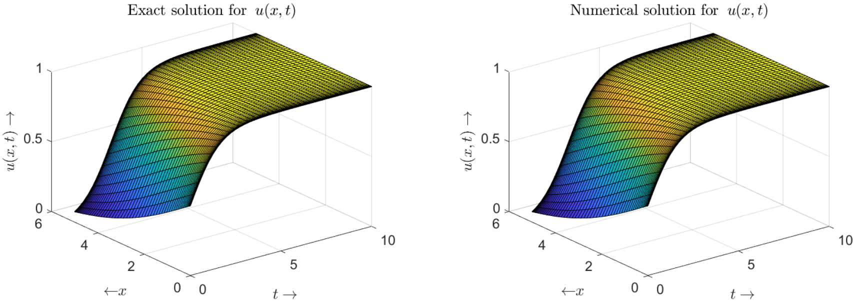


Figure: Exact solution and Numerical solution of Burgers-Fisher equation

Conclusion

- It was discovered in this study that MD-BSQLM generates more computationally efficient results than BHM.
- Further to that, evidence suggests that a larger number of nonoverlapping intervals leads to greater precision and quicker time to generate solutions when using the MD-BSQLM.
- It is evident that the MD-BSQLM is a very effective and also efficient approach for solving broad classes of problems such as the ones examined.

Bibliography

V. Chandraker, A. Awasthi, S. Jayaraj, Numerical treatment of Burger-Fisher equation, Procedia Technology 25 (2016) 1217–1225.

S. S. Nourazar, M. Soori and A. Nazari-Golshan. (2015), On the Exact Solution of Burgers-Huxley Equation Using the Homotopy Perturbation Method, Journal of Applied Mathematics and Physics, Vol. 3, pp. 285–294.

H. Muzara, S. Shateyi, and G.T. Marewo. (2018), On the bivariate spectral quasi-linearization method for solving the two-dimensional Bratu problem, De Gruyter, Vol. 16, pp. 554–562

S.S. Motsa. (2014), On the Bivariate Spectral Homotopy Analysis Method Approach for Solving Nonlinear Evolution Partial Differential Equations, Abstract and Applied Analysis, Vol. 2014, No. 350529.

W. Ibrahim. (2020), Spectral Quasilinearization Method for Solution of Convective Heating Condition, Engineering Transactions, Vol. 68, No. 1, pp. 69–87.

J. A. Kwanamu, Y. Skwame and J. Sabo. (2021), Block Hybrid method for solving higher order ordinary differential equations using power series on implicit one-step second derivative, FUW Trends in Science and Technology

Bibliography

- M.I. Modebei, S. Jator and H. Ramos. (2020), Block Hybrid Method for the Numerical solution of Fourth order Boundary Value Problems, Journal of Computational and Applied Mathematics, Vol. 377, No. 112876.
- I. Hashim, M.S.M. Noorani and M.R.S. Al-Hadidi. (2005), Solving the generalized Burgers-Huxley equation using the Adomian decomposition method, Elsevier Ltd, Vol. 43, pp. 1404–1411.
- R.C. Mittal and A. Tripathi. (2015), Numerical solutions of generalized Burgers–Fisher and generalized Burgers–Huxley equations using collocation of cubic B-splines, International Journal of Computer Mathematics, Vol. 92, No. 5, 1053–1077.
- M. Farhan, Z. Omar, F. Mebarek-Oudina, J. Raza, Z. Shah, R.V. Choudhari and O.D. Makinde. (2020), Implementation of the one-step one hybrid-block method on the nonlinear equation of a circular sector oscillator, Computational Mathematics and Modelling, Vol. 31, No. 1.
- N. Kumar and S. Singh. (2016), Numerical Solution of Burgers-Huxley equation using improved Nodal integral method, Department of Energy Science and Engineering, Indian Institute of Technology.

Thank you