On modular quasi-metric spaces

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Overview

Introduction

The concept of modular metric spaces was introduced by V.V Chistyakov [2] in 2010. The author presented a complete description of generators of Lipschitz continous, bounded and some other classes of superposition operators. Several extensions to his findings then followed. One such study is by Abdou [1] where he investigated 1-local retracts in modular metric spaces with focus on the existence of common fixed points of modular nonexpansive mappings. In his PhD thesis, Sebogodi [4] also extended the results of modular metric spaces to the asymmetric setting where he introduced the concept of Isbell convexity in modular quasi-metric spaces and presented some fixed point theorems.

Modular quasi-metric spaces

In this section, we define concepts in modular quasi-metric spaces.

Definition

Let X be a set. A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular quasi-pseudometric on X if the following conditions are satisfied:

(i) $w(\lambda, x, x) = 0$ *whenever* $x \in X$ *and* $\lambda \in (0, \infty)$,

(ii) $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y)$ *whenever* $x, y, z \in X$ *and* $\lambda, \mu \in (0, \infty)$.

Example

Let $X = \mathbb{R}$. *Define* $w : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty]$ *by*

$$w(\lambda, x, y) = \begin{cases} \infty, & \text{if } x > y \\ 0, & \text{otherwise} \end{cases}$$

whenever $\lambda > 0$. *Then* w *is a modular quasi-metric on* $\mathbb{R}$.

For a modular quasi-pseudometric w on a set X, the function $w^t(\lambda, x, y) : (0, \infty) \times X \times X \to [0, \infty]$ defined by

$$w^t(\lambda, x, y) = w(\lambda, y, x) \ \forall x, y \in X \quad and \quad \lambda \in (0, \infty)$$

is also a modular quasi-pseudometric on X, called the transpose modular quasi-pseudometric of w.

Moreover, it should also be noted that for any modular quasi-pseudometric w on X, the function

$$w^s(\lambda, x, y) = \max\left\{w(\lambda, x, y), w^t(\lambda, y, x)\right\}$$

for all $x, y \in X$ and $\lambda \in (0, \infty)$ is a modular pseudometric on X in the sense of Chistyakov.

For any modular quasi-pseudometric w on a set X, if $w = w^t$, then w is a modular pseudometric on X.
On a set X endowed with a modular quasi-pseudometric w, we have

$$w(\lambda, x, y) \leq w^s(\lambda, x, y) \tag{1}$$

$$w^t(\lambda, x, y) \leq w^s(\lambda, x, y) \tag{2}$$

whenever $\lambda > 0$ and $x, y \in X$.

Furthermore, for any $x \in X$, the set $X_w(x)$ is defined as follows:

$$X_w(x) = \left\{ y \in X : \lim_{\lambda \to \infty} w(\lambda, x, y) = 0 = \lim_{\lambda \to \infty} w^t(\lambda, x, y) \right\}.$$

The set $X_w(x)$ is called a $w-$modular set.
Let us consider an element $x_0 \in X$. The set

$$X_w^*(x_0) = \left\{ x_0 \in X : w(\lambda, x, x_0) < \infty \quad and \quad w(\lambda, x_0, x) < \infty \right\}$$

for some $\lambda > 0$.

The set $X_w^*(x_0)$ is also referred to as a $w-$modular set (around x_0) and x_0 is called the center of X_w^*.
Moreover, the function q_w defined by

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$$

for all $x, y \in X_w$, is a quasi-pseudometric on X_w, whenever w is modular quasi-pseudometric on X.
Note that

$$q_{w^t}(x, y) = q_w(y, x) = (q_w)^t(x, y)$$

for all $x, y \in X_w$.

For any $x \in X_w$ and $\lambda, \mu > 0$, we define the sets $B^w_{\lambda,\mu}(x)$ and $C^w_{\lambda,\mu}(x)$ by

$$B^w_{\lambda,\mu}(x) = \{z \in X_w : w(\lambda, x, z) < \mu\}$$

and

$$C^w_{\lambda,\mu}(x) = \{z \in X_w : w(\lambda, x, z) \leq \mu\}.$$

The set $B^w_{\lambda,\mu}(x)$ is called a w $<$-entourage about x relative to λ and μ, and the set $C^w_{\lambda,\mu}(x)$ is called a w $\leq$-entourage about x relative to λ and μ.

Remark

Let w be a modular quasi-pseudometric on a set X. Then

$$B^{w^s}_{\lambda,\mu}(x) \subseteq B^{w}_{\lambda,\mu}(x)$$

and

$$C^{w^s}_{\lambda,\mu}(x) \subseteq C^{w}_{\lambda,\mu}(x)$$

whenever $x, y \in X_w$ and $\lambda, \mu > 0$.

Definition (V.V. Chistyakov)

Let w be a modular quasi-pseudometric on a set X. Given $x, y \in X$,

(i) *w is said to be continuous from the right on $(0, \infty)$ if for any $\lambda > 0$ we have*

$$w(\lambda, x, y) = w_{+0}(\lambda, x, y).$$

(ii) *w is said to be continuous from the left on $(0, \infty)$ if for any $\lambda > 0$ we have*

$$w(\lambda, x, y) = w_{-0}(\lambda, x, y).$$

(iii) *w is said to be continuous on $(0, \infty)$ if w is continuous from the right and continuous from the left on $(0, \infty)$.*

Remark

If w is continuous from the right on $(0, \infty)$, then for any $x, y \in X_w$ and $\lambda > 0$ we have that $q_w(x, y) \leq \lambda$ if and only if $w(\lambda, x, y) \leq \lambda$.

Normality on modular quasi-metric spaces

Definition

Let w be a modular quasi-pseudometric on X. A nonempty subset A of X_w is said to be w-bounded if there exists $x \in X_w$ such that $A \subseteq C^{w}_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(x)$ for some $\lambda, \mu > 0$.

Remark

Let w be a modular quasi-pseudometric on a set X, then boundedness on (X_w, q_w) implies w-boundedness. This observation follows from the fact that $C_{q_w}(x, \lambda) \subseteq C^{w}_{\lambda,\lambda}(x)$ and $C_{(q_w)^t}(x, \lambda) \subseteq C^{w^t}_{\lambda,\lambda}(x)$ whenever $\lambda > 0$ and $x \in X_w$.

Definition

Let A be a w-bounded subset of X_w. The diameter of A, denoted by $diam_w(A)$, is defined by

$$diam_w(A) = \sup\{w(\lambda, x, y) : x, y \in A\}$$

for some $\lambda > 0$.

Lemma

Let w be a modular quasi-pseudometric on X and A be a subset of X_w. Then $diam_w(A) \leq diam_{q_w}(A)$.

Lemma

Let w be a modular quasi-pseudometric on X. If A is a w-bounded subset of X_w, then $diam_w(A) < \infty$.

Lemma

Let w be a modular quasi-pseudometric on X. If w is continuous from the right on $(0, \infty)$, then boundedness on (X_w, q_w) is equivalent to w-boundedness.

For a w-bounded subset $A \subset X_w$, we set

$$cov(A)_w = \bigcap \left\{ C^w_{\lambda,\lambda}(x) : A \subseteq C^w_{\lambda,\lambda}(x), x \in X_w, \lambda > 0 \right\} \qquad (3)$$

and

$$cov(A)_{w^t} = \bigcap \left\{ C^{w^t}_{\mu,\mu}(x) : A \subset C^{w^t}_{\mu,\mu}(x), x \in X_w, \mu > 0 \right\} \qquad (4)$$

Furthermore, we define the $w - bicover$ of A by

$$bicov_w(A) = cov(A)_w \cap cov(A)_{w^t}.$$

Definition

Let w be a modular quasi-pseudometric on X. A nonempty and w-bounded subset A of X_w is called ***w-admissible*** *if $A = bicov_w(A)$.*

Remark

Note that a w-admissible subset of X_w can be written as the intersection of a family of the form $C^w_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(x)$, where $x \in X_w$ and $\lambda, \mu > 0$.

It should be observed that the collection of all w-admissible subsets of X_w will be denoted by $\mathcal{A}_w(X_w)$.

Lemma

Let w be a modular quasi-pseudometric on X which is continuous from the right on $(0, \infty)$. Then

$$C_{q_w}(x, \lambda) = C^{w}_{\lambda,\lambda}(x)$$

and

$$C_{(q_w)^t}(x, \lambda) = C^{w^t}_{\lambda,\lambda}(x)$$

whenever $\lambda > 0$ and $x \in X_w$.

Corollary

Let w be a modular quasi-pseudometric on X which is continuous from the right on $(0, \infty)$ and $A \subseteq X_w$. Then A is w-admissibe if and only if A is q_w-admissible.

Definition

Let w be a modular quasi-metric on X. We say that:

(i) *The collection $\mathcal{A}_w(X_w)$ is compact if every descending chain of nonempty subsets of $\mathcal{A}_w(X_w)$ has a nonempty intersection.*

(ii) *The collection $\mathcal{A}_w(X_w)$ is normal (or has a normal structure) if for any $A \in \mathcal{A}_w(X_w)$ with A having more than one point, there exists $\lambda > 0, \mu > 0$ such that $\lambda < diam_w(A)$ and $\mu < diam_w(A)$ and for $a \in A$ with $A \subseteq C^{w}_{\lambda,\lambda}(a) \cap C^{w^t}_{\mu,\mu}(a)$.*

Lemma

Let w be a modular quasi-metric on X. Then

(i) *If $\mathcal{A}_w(X_w)$ is compact, then $\mathcal{A}_{q_w}(X_w)$ is compact.*

(ii) *If $\mathcal{A}_w(X_w)$ is normal, then $\mathcal{A}_{q_w}(X_w)$ is normal.*

Theorem

Let w be a modular quasi-metric on X. If X_w is q_w-bounded and $T : X_w \to X_w$ is a w-nonexpansive map, then T has at least one fixed point whenever $\mathcal{A}_w(X_w)$ is compact and normal.

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Thank you.

Dankie.