Locally connected categories

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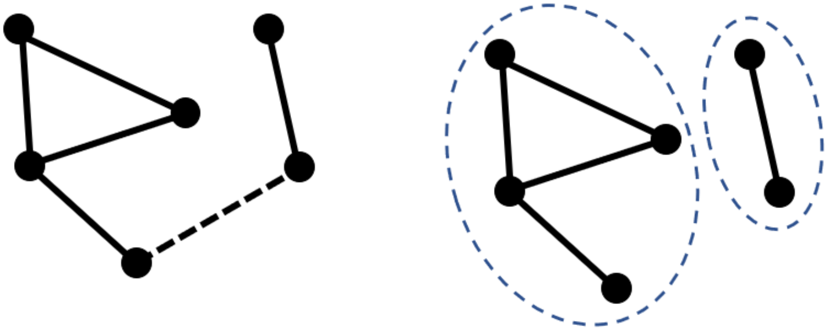


Lextensivity

Definition

Let $\mathbb{C}$ be a category with finite coproducts and finite limits. $\mathbb{C}$ is said to be *lextensive* if for every two objects A and B in $\mathbb{C}$, the coproduct functor

$$+ : \ (\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow A + B)$$

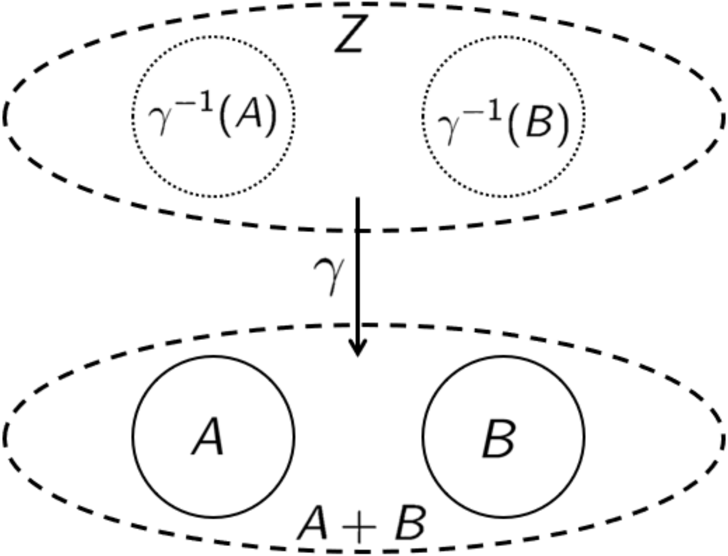
is an equivalence of categories.

Definition

Let $\mathbb{C}$ be a category with finite limits. $\mathbb{C}$ is said to be *infinitary lextensive* if, for every family of objects $(A_i)_{i \in I}$ in $\mathbb{C}$, the coproduct $\sum_{i \in I} A_i$ exists and the functor

$$\sum_{i \in I} : \ \prod_{i \in I} (\mathbb{C} \downarrow A_i) \to (\mathbb{C} \downarrow \sum_{i \in I} A_i)$$

is a category equivalence.


Z
γ⁻¹(A)
γ⁻¹(B)
γ
A
B
A + B

Connected objects

Definition

An object C in a lextensive category $\mathbb{C}$ is said to be *connected* when any of the following equivalent definitions hold:

1. $C \neq 0$, and if $X \to C \leftarrow Y$ is a coproduct diagram, then either $X = 0$ or $Y = 0$.
2. $C \neq 0$, and if $X \to C \leftarrow Y$ is a coproduct diagram, then either $X \to C$ or $Y \to C$ is an isomorphism.
3. Any morphism from C to a coproduct $A + B$ factors through exactly one of the two coproduct injections $A \to A + B$ and $B \to A + B$.
4. The functor $\mathrm{Hom}(C, -) : \mathbb{C} \to \mathit{Set}$ preserves binary (finite) coproducts.

Let $\mathbb{A}$ be an arbitrary category. We define the category $\mathrm{Fam}(\mathbb{A})$ of families of objects in $\mathbb{A}$.

$\mathrm{Fam}(\mathbb{A})$ (finite version denoted $\mathrm{FinFam}(\mathbb{A})$)

Objects: families $(A_i)_{i \in I}$ of objects in $\mathbb{A}$
Morphisms: (f, α), where $f : I \to J$ is a function and $\alpha = (\alpha_i : A_i \to B_{f(i)})_{i \in I}$ is a family of morphisms in $\mathbb{A}$.
Composition of morphisms

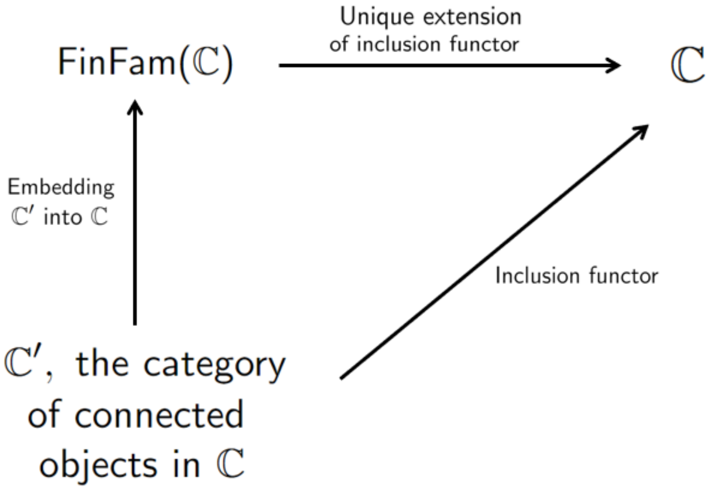
$$(A_i)_{i \in I} \xrightarrow{(f,\alpha)} (B_j)_{j \in J} \xrightarrow{(g,\beta)} (C_k)_{k \in K}$$

defined by

$$(g, \beta)(f, \alpha) = (gf, (\beta_{f(i)}\alpha_i : A_i \to C_{gf(i)})_{i \in I})$$

Considering elements of $\mathbb{A}$ as one-object families gives a full embedding

$$\mathbb{A} \hookrightarrow \mathrm{Fam}(\mathbb{A}).$$


FinFam($\mathbb{C}$)
Unique extension
of inclusion functor
$\mathbb{C}$
Embedding
$\mathbb{C}'$ into $\mathbb{C}$
Inclusion functor
$\mathbb{C}'$, the category
of connected
objects in $\mathbb{C}$

Locally connected

Definition

A category $\mathbb{C}$ with finite limits is said to be *locally connected* if it is infinitary lextensive, and every object in $\mathbb{C}$ can be presented as a coproduct of connected objects.

Examples

Examples

Set, Cat, Preord

Nonexamples

Top (consider the space $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ with the subspace topology from $\mathbb{R}$)

Theorem

Let $\mathbb{C}$ be a category with finite limits. Then the following conditions are equivalent:

1. *$\mathbb{C}$ is locally connected;*
2. *$\mathbb{C}$ is equivalent to Fam($\mathbb{A}$) for some $\mathbb{A}$, and any such equivalence induces an equivalence between $\mathbb{A}$ and the category of connected objects in $\mathbb{C}$.*

References

G. Janelidze, *Reflections and generalized connectedness*, notes based on seminar talks given at University of Cape Town (2006)