Positive Weighted Koopman Semigroups on Banach lattice-modules

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Overview

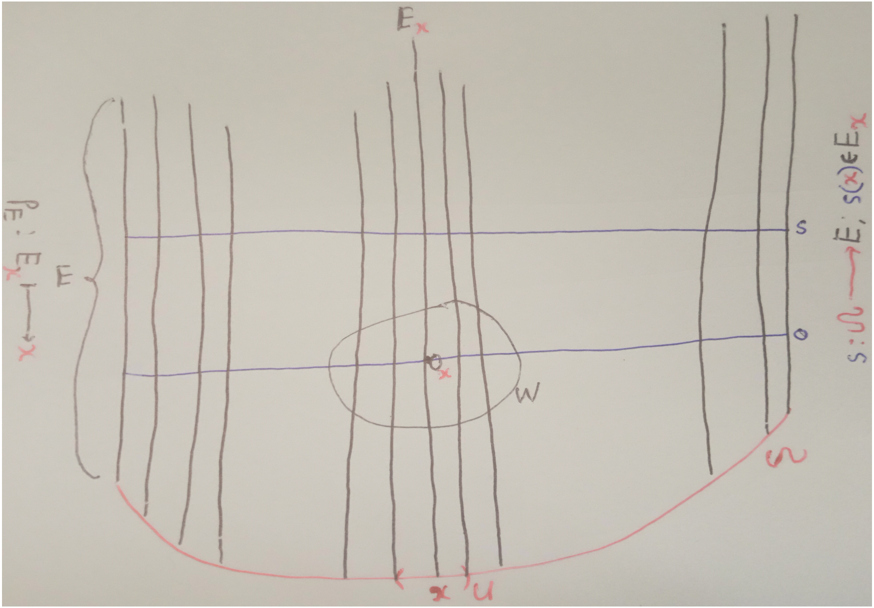
Banach lattice-bundle

Let E be a topological space (*total space*), Ω a locally compact space (*base space*), and $p_E : E \longrightarrow \Omega$ a continuous, open, and surjective mapping (*bundle projection*).

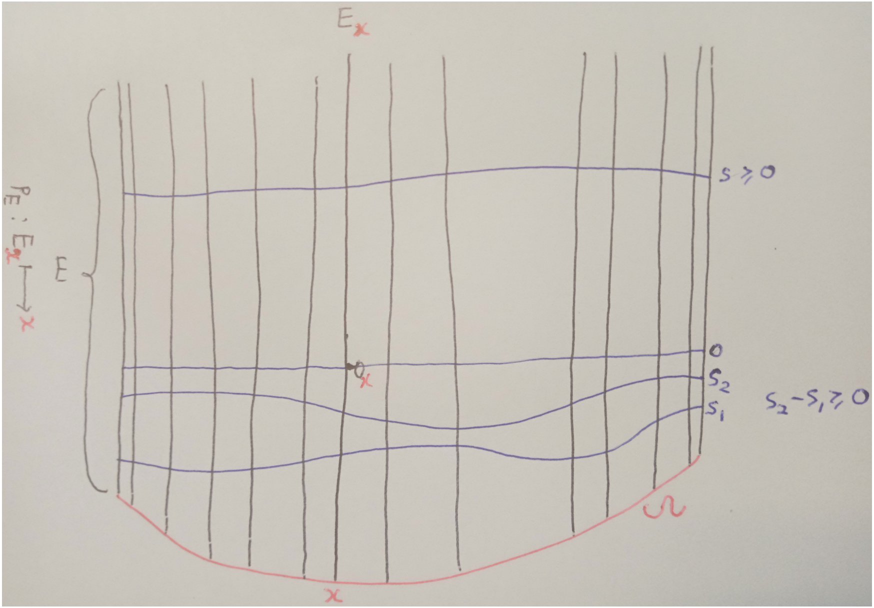
	Banach bundle E **over** Ω	**Banach lattice-bundle** E **over** Ω
(i)	fibers $E_x := p_E^{-1}(x)$ are Banach spaces.	fibers are Banach lattices.
(ii)	$+ : E \times_\Omega E \longrightarrow E,\ (u, v) \mapsto u +_{E_{p_E(v)}} v$ $\cdot : \mathbb{K} \times E \longrightarrow E,\quad (\lambda, v) \mapsto \lambda ._{E_{p_E(v)}} v$	where $E \times_\Omega E := \cup_{x\in\Omega} E_x \times E_x$; and are continuous.
(iii)	$\|\|\cdot\|\| : E \longrightarrow \mathbb{R}_+,\ v \mapsto \|\|v\|\|_{E_{p_E(v)}}$ is upper	semicontinuous (bundle norm).
(iv)	—	$\|\cdot\| : E \longrightarrow E,\quad v \mapsto \|v\|_{E_{p_E(v)}}$ is continuous (bundle modulus).
(v)	For each $x \in \Omega$ and each open set $W \subseteq E$ there are $\varepsilon > 0$ and open set $U \subseteq \Omega$ with	containing zero $0_x \in E_x$ $\left\{v \in p_E^{-1}(U) \mid \|\|v\|\|_{E_{p_E(v)}} \leq \varepsilon\right\} \subseteq W$.
	e.g., $E := \Omega \times Z$; Z a Banach space.	$E := \Omega \times Z$; Z a Banach lattice.
	$\Gamma_0(\Omega, E) := \{s : \Omega \longrightarrow E \mid p_E \circ s = Id_\Omega$ s continuously vanish at "infinity"$\}$ $\|\|s\|\| := \sup_{x\in\Omega} \|\|s(x)\|\|_{E_x}$; AM-module	Becomes AM m-lattice-module. $\|s\| : \Omega \longrightarrow E; x \longrightarrow \|s(x)\|_{E_x}$

If the bundle norm is continuous, then E is a continuous *topological* Banach lattice-bundle over Ω.

Banach bundle



Banach lattice-bundle



Banach lattice-module

Let A be a commutative Banach algebra. And L a commutative Banach lattice-algebra.

	Banach module Γ over A	**Banach lattice-module Γ over L**
(i)	Γ is a Banach space	Is a Banach lattice
(ii)	$A \times \Gamma \longrightarrow \Gamma; (a, s) \mapsto a \cdot s$ is a module	the same
(iii)	$\|\|a \cdot s\|\|_\Gamma \leq \|\|a\|\|_A \|\|s\|\|_\Gamma; a \in A, s \in \Gamma$	the same
(iv)	—	$\|f \cdot s\|_\Gamma \leq \|f\|_L \cdot \|s\|_\Gamma; f \in L, s \in \Gamma$
	e.g., Banach algebra A; $A \times A \longrightarrow A; (a, b) \mapsto a \star b$	Banach lattice-algebra L; $\|f \star g\| \leq \|f\| \star \|g\|; f, g \in L$
	$\Gamma_0(\Omega, E)$ over $C_0(\Omega)$; E Banach bundle	

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(v)	—	If $\|f \cdot s\|_\Gamma = \|f\|_L \cdot \|s\|_\Gamma; f \in L, s \in \Gamma$; we call it an m-Banach lattice-module e.g., $\Gamma_0(\Omega, E)$ over $C_0(\Omega)$; E Banach lattice-bundle
	submodules	lattice-submodules; ideal-submodules

Gelfand and Kakutani Representation theorems

Theorem (Gelfand's [3])

Let A be a commutative C^*-algebra[a]. Then, there exists a unique (up to homeomorphism) compact space[b] K such that $A \longrightarrow C(K)$ is an isometric $*$-algebra isomorphism.

[a] with unit

[b] the character space of A.

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Theorem (Kakutani's [5])

Let L be an AM-space[a] with unit[b]. Then, there exists a unique (up to homeomorphism) compact space[c] K such that $L \longrightarrow C(K)$ is an isometric lattice isomorphism.

[a] $||f_1 \vee f_2|| = \max\{||f_1||, ||f_2||\}\ \forall f_1, f_2 \in L_+$;
[b] $L = L_u := \{f \in L : |f| \leq \lambda u, \lambda > 0, u \in L_+ \text{fixed}\}$.
[c] the Šilov boundary of L.

Semigroup Representations

Let G be a locally compact group, and S a closed subsemigroup of G containing the neutral element e, i.e., a closed 'submonoid' of G.

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(Strongly Continuous) Semigroup Representations

For a Banach space X, a monoid representation[a]

$$T : S \longrightarrow \mathscr{L}(X);\; t \mapsto T(t)$$

is called strongly continuous if the mapping

$$S \longrightarrow X;\; t \mapsto T(t)x$$

is continuous for each $x \in X$. And we call $T = (T(t))_{t \in S}$ a strongly continuous semigroup representation on X.

[a] i.e., $T(t_1 t_2) = T(t_1)T(t_2)\ \forall\ t_1, t_2 \in S$ and $T(e) = Id_X$.

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Note: If $G = \mathbb{R}$, S $= \mathbb{R}_{\geq 0}$, then $(T(t))_{t \geq 0}$ is a C_0-semigroup on X.

Topological versus C^*-algebra G-dynamical systems

Gelfand's theorem ascertains a contravariant equivalence of categories

$$\left\{\varphi : G \xrightarrow[t\mapsto\varphi_t]{cont} Aut(K)\right\} \mapsto \left\{\boldsymbol{T}_{\boldsymbol{\varphi}} : G \xrightarrow[t\mapsto T_{\varphi_t}]{strongly\ cont} Aut(C(K))\right\}$$

$$\theta \mapsto V_\theta$$

between the category of *topological* G-dynamical systems[1] and the category of commutative G-dynamical C^*-algebras[2].

[1] denoted as $(K, (\varphi_t)_{t\in G})$; and $(\varphi_t)_{t\in G}$ is called a *continuous* flow on K.

[2] denoted as $(C(K), \boldsymbol{T}_{\boldsymbol{\varphi}})$; the Koopman group representation on $C(K)$.

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For each $t \in G$;

$$\begin{tikzcd} K_1 \arrow[r, "\theta"] \arrow[d, "\varphi_t"'] & K_2 \arrow[d, "\psi_t"] \\ K_1 \arrow[r, "\theta"'] & K_2 \end{tikzcd} \iff \begin{tikzcd} C(K_1) \arrow[d, "T_{\varphi_t} f = f \circ \varphi_{t^{-1}}"'] & C(K_2) \arrow[l, "f \circ \theta \overset{V_\theta}{\longleftarrow} f"'] \arrow[d, "T_{\psi_t} g = g \circ \psi_{t^{-1}}"] \\ C(K_1) & C(K_2) \arrow[l, "V_\theta"] \end{tikzcd}$$

[1]denoted as $(K, (\varphi_t)_{t\in G})$; and $(\varphi_t)_{t\in G}$ is called a *continuous* flow on K.

[2]denoted as $(C(K), \boldsymbol{T}_{\boldsymbol{\varphi}})$; the Koopman group representation on $C(K)$.

Topological versus AM-space G-dynamical systems

Kakutani's theorem also ascertains a contravariant equivalence of categories

$$\left\{\varphi : G \xrightarrow[t\mapsto\varphi_t]{cont} Aut(K)\right\} \mapsto \left\{\boldsymbol{T}_\varphi : G \xrightarrow[t\mapsto T_{\varphi_t}]{strongly, cont} Aut(C(K))\right\}$$

$$\theta \mapsto V_\theta$$

between the category of *topological* G-dynamical systems[3] and the category of G-dynamical AM-spaces[4] with units[5].

[3] (on compact spaces)
[4] $||f_1 \vee f_2|| = \max\{||f_1||, ||f_2||\}\ \forall f_1, f_2 \in L_+$;
[5] $L = L_u := \{f \in L : |f| \leq \lambda u, \lambda > 0, u \in L_+ \text{fixed}\}$

GELFAND-TYPE THEOREMS FOR DYNAMICAL BANACH MODULES

HENRIK KREIDLER AND SITA SIEWERT

arXiv:1903.05035v1 [math.DS] 12 Mar 2019

ABSTRACT. The representation theorems of Gelfand and Kakutani for commutative C*-algebras and AM- and AL-spaces are the basis for the Koopman linearization of topological and measure-preserving dynamical systems. In this article we prove versions of these results for dynamics on topological and measurable Banach bundles and the corresponding weighted Koopman representations on Banach modules.

Mathematics Subject Classification (2010). 46L08, 46M15, 47A67, 47D03.

1. INTRODUCTION

The concept of *Koopman linearization* provides a very powerful method to study dynamical systems, see [EFHN15]. Given a *topological G-dynamical system*, i.e., a locally compact group G acting continuously on a locally compact space Ω, one can consider the induced *Koopman representation* of G as automorphisms of the commutative C*-algebra $\mathrm{C}_0(\Omega)$ of all continuous functions on Ω vanishing at infinity given by $T(g)f(x) := f(g^{-1}x)$ for $x \in \Omega$, $f \in \mathrm{C}_0(\Omega)$, and $g \in G$.

Example (from Cocycles and Skew-products)

Consider a *continuous* cocycle $\{\phi(x) \in \mathscr{L}(Z) : x \in \Omega\}$ associated to a homeomorphism $\varphi : \Omega \longrightarrow \Omega$ on a locally compact space Ω with Banach space Z, i.e.,

$$\begin{cases} \phi^0(x) = Id_Z \text{ for } x \in \Omega; \\ \phi^{n+m}(x) = \phi^n(\varphi^m(x))\phi^m(x) \text{ for } x \in \Omega \text{ and } n, m \in \mathbb{N}; \text{ and} \\ \Omega \times Z \longrightarrow Z; (x, v) \mapsto \phi(x)v \text{ is continuous for all } v \in Z, \end{cases}$$

and the Koopman operator $T_\varphi : C_0(\Omega) \longrightarrow C_0(\Omega); f \mapsto f \circ \varphi^{-1}$.

Then, the *continuous* linear skew-product

$$\Phi : E \longrightarrow E; (x, v) \mapsto (\varphi(x), \phi(x)v) \qquad \text{where} \quad E := \Omega \times Z$$

induces a weighted Koopman operator $\mathcal{T}_\Phi : C_0(\Omega, Z) \longrightarrow C_0(\Omega, Z); s \mapsto \Phi \circ s \circ \varphi^{-1}$;

i.e., $\mathcal{T}_\Phi \in \mathscr{L}(C_0(\Omega, Z))$ and $\mathcal{T}_\Phi fs = T_\varphi f \mathcal{T}_\Phi s$ for $f \in C_0(\Omega)$ and $s \in C_0(\Omega, Z)$ with $(fs)(x) := f(x)s(x)$ for all $x \in \Omega$.

Gelfand-type theorem for dynamical Banach modules: Motivation

The notion of a cocycle over a *continuous* flow is useful:

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(i) in modelling non-autonomous problems, e.g., dissipative partial differential equations, in particular, the Navier-Stokes equation;

(ii) as an abstract framework for the study of random dynamical systems;

(iii) in the general theory of ideal fluid dynamics;

(iv) in detecting the existence of the so-called exponential dichotomy(or hyperbolicity); and

(v) in the generalisation of the classical notion of two-parameter evolution family.

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Note: Weighted Koopman operators are also known in the literature as weighted composition or weighted shift operators in general operator theory, as transfer operators in dynamical systems theory, and as push-forward operators in the theory of differentiable manifolds.

Theorem (H. Kreidler and S. Siewert 's [4])

Let G be a locally compact group, $S \subseteq G$ a closed submonoid, and $(\Omega, (\varphi_t)_{t\in G})$ a topological G-dynamical system. Then the assignments

$$(E, \mathbf{\Phi}) \longmapsto (\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical topological Banach bundles over $(\Omega, (\varphi_t)_{t\in G})$ to the category of S-dynamical AM-modules[a] over $(C_0(\Omega), \boldsymbol{T}_\varphi)$.

[a] locally convex $C_0(\Omega)$-modules

Gelfand-type theorem for dynamical Banach lattice-modules

Theorem (David's thesis, 2022)

Let G be a locally compact group, $S \subseteq G$ a closed submonoid, and $(\Omega, (\varphi_t)_{t\in G})$ a topological G-dynamical system. Then the assignments

$$(E, \mathbf{\Phi}) \longmapsto (\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of positive S-dynamical topological Banach lattice-bundles over $(\Omega, (\varphi_t)_{t\in G})$ to the category of S-dynamical AM m-lattice-modules[a] over $(C_0(\Omega), \boldsymbol{T}_\varphi)$.

[a] locally convex $C_0(\Omega)$-m-lattice-modules

Gelfand-type theorem for dynamical Banach lattice-modules: Positive weighted semigroup

Let $\mathbf{T} : G \longrightarrow Aut(L); t \mapsto T_t$ be a strongly continuous group representation on a commutative Banach lattice-algebra L, denoted as $(L, \mathbf{T})$.

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A positive S-dynamical Banach lattice-module over $(L, \mathbf{T})$ is a pair $(\Gamma, \boldsymbol{\mathcal{T}})$ consisting of Γ a Banach lattice-module over L and a monoid representation

$$\boldsymbol{\mathcal{T}} : S \longrightarrow \mathscr{L}(\Gamma);\ t \mapsto \mathcal{T}(t)$$

such that:

(i) $\mathcal{T}(t)$ is positive T_t-homomorphism for each $t \in S$, i.e., $\mathcal{T}(t)$ is positive and $\mathcal{T}(t)fs = T_t f \mathcal{T}(t)s$ for every $f \in L, s \in \Gamma$;

(ii) $\boldsymbol{\mathcal{T}}$ is strongly continuous, i.e.,

$$S \longrightarrow \Gamma,\ t \mapsto \mathcal{T}(t)s$$

is continuous for every $s \in \Gamma$.

We call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(t))_{t \in S}$ a positive weighted semigroup representation on Γ over $(L, \mathbf{T})$ (or over $\mathbf{T} = (T_t)_{t \in G}$ on L).

Main theorem: Gelfand-type theorem for dynamical Banach lattice-modules

Theorem (David's thesis, 2022)

Let Γ be an AM m-lattice-module over $C_0(\Omega)$. Then, any positive weighted semigroup representation $\mathcal{T} = (\mathcal{T}(t))_{t\in S}$ on Γ over $(C_0(\Omega), \boldsymbol{T}_\varphi)$ is (unique up to isometric isomorphy) a positive weighted Koopman semigroup representation over $(C_0(\Omega), \boldsymbol{T}_\varphi)$.

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Precisely, there exists a unique (up to isometric isomorphy) pair $(E, \boldsymbol{\Phi})$ of positive S-dynamical topological Banach lattice-bundle over $(\Omega, (\varphi_t)_{t\in G})$ such that:

$$(\mathcal{T}(t))_{t\in S} \cong (\mathcal{T}_\Phi(t))_{t\in S} \quad \text{on} \quad \Gamma \cong \Gamma_0(\Omega, E).$$

Moreover, $\boldsymbol{\mathcal{T}} = (\mathcal{T}(t))_{t\in S}$ is a positive-isometry if and only if $\boldsymbol{\Phi} = (\Phi_t)_{t\in S}$ is a positive-isometry.

Gelfand-type theorem for dynamical Banach modules: AM-module

Theorem (H. Kreidler and S. Siewert 's [4])

Let Γ be a Banach module over commutative Banach algebra $C_0(\Omega)$. Then the following are equivalent:

(i) *Γ is an AM-module.*

(ii) $||(f_1 \vee f_2)s|| = \max(||f_1 s||, ||f_2 s||)$ *for all* $f_2, f_2 \in C_0(\Omega)_+$ *and* $s \in \Gamma$.

(iii) *Γ is isometrically isomorphic to the Banach space $\Gamma_0(\Omega, E)$ of continuous sections vanishing at "infinity" of a (unique up to isometric isomorphy) topological Banach bundle $p_E : E \longrightarrow \Omega$.*

Rough sketch of the proof: Let $E := \dot{\bigcup}_{x\in\Omega} E_x$ for each $x \in \Omega$, identifying the Banach space $E_x := \Gamma / J_x$ with

$$J_x := \overline{lin}\{fs : f \in C_0(\Omega) \text{ with } f(x) = 0 \text{ and } s \in \Gamma\}.$$

Gelfand-type theorem for dynamical Banach lattice-modules: AM m-lattice-module

Theorem (David's thesis, 2022)

Let Γ be an m-Banach lattice-module over commutative Banach lattice-algebra $C_0(\Omega)$. Then the following are equivalent:

(i) *Γ is an AM m-lattice-module.*

(ii) $||(f_1 \vee f_2)s|| = \max(||f_1 s||, ||f_2 s||)$ *for all* $f_1, f_2 \in C_0(\Omega)_+$ *and* $s \in \Gamma$.

(iii) *Γ is isometrically isomorphic to the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at "infinity" of a (unique up to isometric isomorphy) topological Banach lattice-bundle* $p_E : E \longrightarrow \Omega$.

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Gelfand-type theorem for dynamical Banach lattice-modules

For each $t \in S$;

$$
\begin{tikzcd}
\Omega \arrow[d, "\varphi_t"'] & E \arrow[l, "P_E"'] \arrow[r, "\Theta"] \arrow[d, "\Phi_t"'] & F \arrow[r, "P_F"] \arrow[d, "\Psi_t"] & \Omega \arrow[d, "\varphi_t"] \\
\Omega & E \arrow[l, "P_E"] \arrow[r, "\Theta"'] & F \arrow[r, "P_F"'] & \Omega
\end{tikzcd}
$$

$$\Longleftrightarrow$$

$$
\begin{tikzcd}
\Gamma_0(\Omega, E) \arrow[r, "s \xmapsto{V_\Theta} \Theta \circ s"] \arrow[d, "\mathcal{T}_\Phi(t)s = \Phi_t \circ s \circ \varphi_{t-1}"'] & \Gamma_0(\Omega, F) \arrow[d, "\mathcal{T}_\Psi(t)r = \Psi_t \circ r \circ \varphi_{t-1}"] \\
\Gamma_0(\Omega, E) \arrow[r, "V_\Theta"'] & \Gamma_0(\Omega, F)
\end{tikzcd}
$$

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Positivity

Banach lattice algebras: some questions, but very few answers

A. W. Wickstead[1]

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Abstract We pose a number of questions and problems about Banach lattice algebras. These concern: What should the definition be? How to add an identity. Order theoretic properties of the multiplication. Order theoretic properties of the left regular representation.

Keywords Riesz algebras · Representations · Norms · Banach Lattice algebras

Mathematics Subject Classification 06A70

Banach lattice-algebra ([6])
$(L, |\cdot|, \star)$:
Is a Banach lattice $(L, |\cdot|)$;
a Banach algebra $(L, \star)$
such that
$$|f \star g| \leq |f| \star |g|$$
for all $f, g \in L$.

Conclusion. What is new?

What is now known?

- Every Banach lattice-algebra is a Banach lattice-module *over itself*.

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- In fact, every Banach lattice is an m-Banach lattice-module *over its center*.

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- Every Banach lattice-algebra is a Banach lattice-module *over itself*.
- In fact, every Banach lattice is an m-Banach lattice-module *over its center*.

What is new?

1. There does not seem to be a concise concept of a Banach lattice being a Banach module over a *commutative* Banach lattice-algebra. So, we propose the definition of a Banach lattice-module for our study.
2. More so, our definition can be seen to be the generalisation of certain Banach lattice $L^\infty(\mathcal{G})$-module as defined by K-T. Eisele and S. Taieb, see [2, Definition 5.3(iii), p.531-532].

Thank you for your attention!

Supervisor: Dr Retha Heymann

(*If I have seen further, it is by standing on the shoulders of giants. - Sir Isaac Newton, 1675*)

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