0-Cauchy completions in strong partial b-metric spaces

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December 8, 2022

Outline of the talk

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- Introduction.
- Strong b-metric spaces and partial b-metric spaces.
- Completeness in strong partial b-metric spaces.
- Completions of strong partial b-metric spaces.
- Conclusions.

1.1. Introduction

- In the book

W. Kirk, N. Shahzad, Fixed point theory in distance spaces, Springer,(2014)

Strong b-metric spaces were introduced by Kirk and Shahzad as generalization of metric spaces. And

- In the article.

S. Shukla, Partial b-metric spaces and fixed point theorems. Mediterr. J. 703-711 (2014)

partial b-metric spaces were introduced as generalization of partial metric spaces. In this talk we introduce a new notion, called strong partial b-metric space, which is the generalization of both strong b-metric spaces and partial metric spaces, we will also present 0-Cauchy completions of a strong partial b-metric space. Strong b-metric spaces and partial b-metric spaces, were introduced with the aim of generalizing Banach fixed point theorem.

- In the article

T. Van An. N. Van Dung. Answers to Kirk-Shahzad's questions on strong b-metric spaces, Taiwan.J. Math. 20 (5) (2016) 1175–1184.

The following notion is defined.

Definition

Let X be a nonempty set and $\alpha \geq 1$ a real constant. A map $d\colon X \times X \longrightarrow [\,0, \infty\,)$ is a strong b-metric on X if for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + \alpha d(y, z)$.

The pair (X, d) is called strong b-metric space.

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- In the article

S. Shukla, Partial b-metric spaces and fixed point theorems, Mediterr.J. Math. 11 (5) (2014) 703–711.

The following notion is defined.

Definition

Let X be a nonempty set. A map $d : X \times X \to [0, \infty)$ is called a partial b-metric *on X if for all $x, y, z \in X$ and $\alpha \geq 1$ the following conditions hold:*

(i) $x = y$ if and only if $d(x, x) = d(x, y) = d(y, y)$;

(ii) $d(x, x) \leq d(x, y)$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, y) \leq \alpha [d(x, z) + d(z, y)] - d(z, z)$

The pair (X, d) is called a partial b-metric space.

Cont...

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- We now introduce a new notion called strong partial b-metric space.

- In the article

S.P Moshokoa, F.T Ncongwane On completeness in strong partial b-metric spaces, strong b-metric spaces and 0-Cauchy completions, Topology and its Applications. 275, (2020) 107011.

Definition

Let X *be a nonempty set. A map* $d : X \times X \longrightarrow [0, \infty)$ *is a* strong partial b-metric *on* X *if for all* $x, y, z \in X$, *and* $\alpha \geq 1$ *the following conditions hold:*

(i) $x = y$ *if* $d(x, x) = d(x, y) = d(y, y)$;

(ii) $d(x, x) \leq d(x, y)$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, z) \leq d(x, y) + \alpha d(y, z) - d(y, y)$.

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Remark

Every strong b-metric space is a strong partial b-metric space but the converse is not necessarily true.

Definition

Let (X, d) be a strong partial b-metric space.

(i) *a sequence* $\{x_n\}$ converges *to a point* $x \in X$ *if* $d(x, x) = \lim_n d(x_n, x) = \lim_n d(x_n, x_n)$.

Cont...

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(ii) *a sequence* $\{x_n\}$ *is called* Cauchy *if* $\lim_{n,m} d(x_n, x_m)$ *exists and is finite.*

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(iii) (X, d) *is called* Cauchy complete *if every Cauchy sequence* $\{x_n\}$ *converges to* $x \in X$.

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(iv) *A sequence* $\{x_n\}$ *is called* 0-Cauchy *if* $\lim_{n,m} d(x_n, x_m) = 0$.

Cont...

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Every strong b-metric space is a strong partial b-metric space but the converse is not necessarily true.

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Let (X, d) be a strong partial b-metric space.

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(iii) (X, d) *is called* Cauchy complete *if every Cauchy sequence* $\{x_n\}$ *converges to* $x \in X$.

(iv) *A sequence* $\{x_n\}$ *is called* 0-Cauchy *if* $\lim_{n,m} d(x_n, x_m) = 0$.

(v) (X, d) *is called* 0- Cauchy complete *if every* 0-*Cauchy sequence converges to a point* $x \in X$ *and* $d(x, x) = 0$.

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Remark

(i) *Every* 0*-Cauchy sequence is a Cauchy sequence but the converse is not necessarily true.*

(ii) *Every Cauchy complete strong partial b-metric space is* 0*-Cauchy complete but the converse is not necessarily true.*

Definition

Let (X, d) be a strong partial b-metric space and Y be a subset of X. We say Y is sequentially dense in X if for $x \in X$, there is a sequence $\{y_n\}$ in Y that converges to x.

Definition

Let $T : (X, d_X) \longrightarrow (Y, d_Y)$ be a map between partial b-metric spaces. T is called an isometry if

$$d_Y(Tx, Ty) = d_X(x, y),$$

for all $x, y \in X$.

Definition

Let (X, d) be a strong partial b-metric space. We say that a strong partial b-metric space $(\bar{X}, \bar{d})$ is a 0-Cauchy completion of (X, d) if
(i) $(\bar{X}, \bar{d})$ is 0-Cauchy complete
(ii) $X \subseteq \bar{X}$, and $\bar{d}|_{X \times X = d}$
(iii) there exists $T : (X, d) \longrightarrow (\bar{X}, \bar{d})$, such that T is an isometry;
(iv) TX is sequentially dense in $\bar{X}$.

Cont...

Given a strong partial b-metric space (X, d). Let
$\mathcal{C} = \{x_n : \{x_n\}$ be a 0-Cauchy sequence$\}$. And
$\mathcal{K} =$
$\{x : \{x\}$ is a constant sequence which is not a 0-Cauchy sequence$\}$.
$\sim$ is an equivalent relation on the class of 0-Cauchy sequences $\mathcal{C}$ and on the class of eventually constant sequences $\mathcal{K}$. Let $\bar{X}$ be the set of all equivalent classes in $\mathcal{K}$ together with the set of all equivalent classes in $\mathcal{C}$, that is
$\bar{X} = \{[\{x\}] : x \in \mathcal{K}\} \cup \{[\{x_n\}] : \{x_n\} \in \mathcal{C}\}$.
For every $\bar{x}, \bar{y} \in \bar{X}$, define $\bar{d} : \bar{X} \times \bar{X} \longrightarrow [0, \infty)$ by

$$\bar{d}(\bar{x}, \bar{y}) = \lim_n d(x_n, y_n),$$

where $\bar{x} = [\{x_n\}]$ and $\bar{y} = [\{y_n\}]$.

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Theorem

Every strong partial b-metric space (X, d) *admits a* 0*-Cauchy completion* $(\bar{X}, \bar{d})$.

Proof of summary.

(i) $\bar{d}$ is well defined.

(ii) $(\bar{X}, \bar{d})$ is a strong partial b-metric space.

(iii) $T : X \longrightarrow \bar{X}$ is an isometry.

(iv) TX is a sequentially dense in $\bar{X}$.

(v) $(\bar{X}, \bar{d})$ is 0-Cauchy complete.

□

Theorem

The 0-Cauchy completion of a strong partial b-metric space (X, d), is unique up isometry.

Proof.

Let $(\bar{X}, \bar{d})$ and $\left(\acute{X}, \acute{d}\right)$ be the two 0-Cauchy completions of (X, d). Then there exists isometric embeddings $T_1 : X \longrightarrow \bar{X}$ and $T_2 : X \longrightarrow \acute{X}$. For each $\bar{x} \in \bar{X}$, we can find $\{x_n\}$ in X such that $T_1 x_n$ converges to $\bar{x}$. Also $T_2 x_n$ converges to to some $\acute{x} \in \acute{X}$. Define $\varphi : \bar{X} \longrightarrow \acute{X}$, by $\varphi(\bar{x}) = \acute{x}$. The map φ is bijective and an isometry □

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- If (X, d) is a partial metric space then $(\bar{X}, \bar{d})$ is a 0-Cauchy completion. As obtained in

S.P Moshokoa. On the 0-Cauchy completion of a partial metric space, Turk. J. Math. Comput. Sci. 4(2016) 10–15.

N. Van Dung. On completion of partial metric spaces, Quaestiones Mathematicae. 40:5 (2017) 589–597.

- If (X, d) is a strong b-metric space then $(\bar{X}, \bar{d})$ is a strong b-metric completion. As obtained in

T. Van An. N. Van Dung. Answers to Kirk-Shahzad's questions on strong b-metric spaces, Taiwan.J. Math. 20 (5) (2016) 1175–1184.

- If (X, d) is a metric space then $(\bar{X}, \bar{d})$ is the well known standard metric completion.

References

Further study

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S.P Moshokoa, F.T Ncongwane 2020.
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Thank you for listening