Iterated Function System of Generalized Contractive Operators

Talat Nazir
Department of Mathematical Sciences
University of South Africa

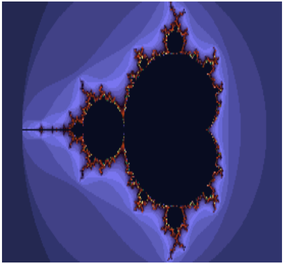
December 6-8, 2022

Fractals:

Fractals are geometrical shape Having mathematical structures defined by two properties:

- Iterative
- Self-similar
  - Zooming in or out on the image reveals deep repetition of patterns





Banach contraction principle

Theorem

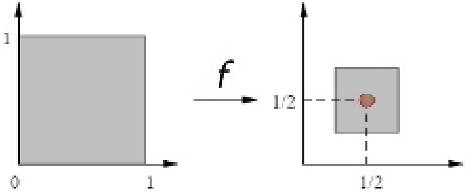
Let (X, d) be a complete metric space and $f : X \to X$ a contraction on X with contraction constant $\alpha \in [0, 1)$, that is, for any $x, y \in X$, the following holds:

$$d(fx, fy) \leq \alpha d(x, y). \tag{1.1}$$

Then f has a unique fixed point in X. Furthermore, for any initial guess $x_0 \in X$ the sequence of simple iterates $\{x_0, fx_0, f^2x_0, f^3x_0, ...\}$ converges to a fixed point of f.

Contraction mappings

Example Let $X = [0,1]^2$ with Euclidean metric d and let $f : X \to X$ be defined by $f(x,y) = (\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{4})$; $(x,y) \in [0,1]^2$.



Let $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2) \in [0,1]^2$, then

$$\begin{aligned} d(f(\mathbf{x}_1), f(\mathbf{y}_2)) &= \sqrt{(x_1^2 + \frac{1}{4} - x_2^2 - \frac{1}{4})^2 + (y_1^2 + \frac{1}{4} - \frac{y_2}{2} - \frac{1}{4})^2} \\ &\leq \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{1}{2} d(\mathbf{x}_1, \mathbf{x}_2). \end{aligned}$$

Hence, f is a contraction with $\alpha = \frac{1}{2}$ and has a unique fixed point $(\frac{1}{2}, \frac{1}{2})$.

Pompeiu-Hausdorff metric

Let (X, d) be a metric space and $\mathcal{H}(X)$ denotes the set of all non-empty compact subsets of X. For $A, B \in \mathcal{H}(X)$, let

$$H(A, B) = \max\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\},$$

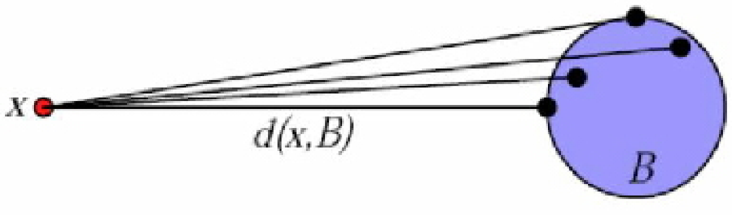
where $d(x, B) = \inf\{d(x, b) : b \in B\}$ is the distance of a point x from the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d.

Pompeiu-Hausdorff metric

Let (X, d) be a metric space and $\mathcal{H}(X)$ denotes the set of all non-empty compact subsets of X. For $A, B \in \mathcal{H}(X)$, let

$$H(A, B) = \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

where $d(x, B) = \inf\{d(x, b) : b \in B\}$ is the distance of a point x from the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d.



Lemma

Lemma

Let (X, d) be a metric space. For all $A, B, C, D \in \mathcal{H}(X)$, the following hold:

1. *If $B \subseteq C$, then* $\sup\limits_{a \in A} d(a, C) \leq \sup\limits_{a \in A} d(a, B)$.

Lemma

Lemma

Let (X, d) *be a metric space. For all* $A, B, C, D \in \mathcal{H}(X)$, *the following hold:*

1. *If* $B \subseteq C$, *then* $\sup_{a \in A} d(a, C) \leq \sup_{a \in A} d(a, B)$.
2. $\sup_{x \in A \cup B} d(x, C) = \max\{\sup_{a \in A} d(a, C), \sup_{b \in B} d(b, C)\}$.

Lemma

Lemma

Let (X, d) be a metric space. For all $A, B, C, D \in \mathcal{H}(X)$, the following hold:

1. *If $B \subseteq C$, then* $\sup\limits_{a\in A} d(a, C) \leq \sup\limits_{a\in A} d(a, B)$.
2. $\sup\limits_{x\in A\cup B} d(x, C) = \max\{\sup\limits_{a\in A} d(a, C), \sup\limits_{b\in B} d(b, C)\}$.
3. $H(A \cup B, C \cup D) \leq \max\{H(A, C), H(B, D)\}$.

Definition

Wardowski [3] introduced a new contraction called F-contraction and proved a fixed point result as an interesting generalization of the Banach contraction principle.
Let $\mathcal{F}$ be the collection of all continuous mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(F_2) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

[3] D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., 2012:94 (2012), 1-6.

F-contractions

Definition

Let (X, d) be a metric space. A self-mapping f on X is called an F-contraction if for any $x, y \in X$, there exists $F \in \digamma$ and $\tau > 0$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)), \tag{1.2}$$

whenever $d(fx, fy) > 0$.

Examples

Following examples show that there are variety of contractive conditions corresponding to different choices of elements in $\mathcal{F}$.

- If we take $F(\lambda) = \ln(\lambda)$ for $\lambda > 0$. Then F satisfies (F_1) to (F_3). A mapping $f : X \to X$ satisfying (1.2) is a contraction with contractive factor $e^{-\tau}$, that is,

$$d(fx, fy) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, \; fx \neq fy.$$

Examples

Following examples show that there are variety of contractive conditions corresponding to different choices of elements in $\mathcal{F}$.

- If we take $F(\lambda) = \ln(\lambda)$ for $\lambda > 0$. Then F satisfies (F_1) to (F_3). A mapping $f : X \to X$ satisfying (1.2) is a contraction with contractive factor $e^{-\tau}$, that is,

$$d(fx, fy) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, \; fx \neq fy.$$

- If we take $F(\lambda) = \ln(\lambda) + \lambda$, $\lambda > 0$, then F satisfies (F_1) to (F_3) and mapping f is of the form

$$d(fx, fy) \leq e^{d(x,y) - d(fx,fy) - \tau} d(x, y), \text{ for all } x, y \in X, \; fx \neq fy.$$

Results of F-Contraction

Theorem

Let (X, d) be a complete metric space and $f : X \to X$ an F-contraction mapping, that is, or any $x, y \in X$, there exists $F \in \digamma$ and $\tau > 0$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)),$$

whenever $d(fx, fy) > 0$. Then f has a unique fixed point in X and for every x_0 in X a sequence of iterates $\{x_0, fx_0, f^2x_0, \ldots\}$ converges to the fixed point of f.

Results

Theorem

Let (X, d) be a metric space and $f : X \to X$ an F-contraction.

1. *Then f maps elements in $\mathcal{H}(X)$ to elements in $\mathcal{H}(X)$.*
2. *If for any $A \in \mathcal{H}(X)$,*

$$f(A) = \{f(x) : x \in A\},$$

then $f : \mathcal{H}(X) \to \mathcal{H}(X)$ is a F-contraction mapping on $(\mathcal{H}(X), H)$.

Result 1

Theorem

Let (X, d) be a metric space and $\{f_n : n = 1, 2, ..., N\}$ a finite family of F-contraction self-mappings on X. Define $T : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$\begin{aligned} T(A) &= f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) \\ &= \cup_{n=1}^{N} f_n(A), \ \textit{for each } A \in \mathcal{H}(X). \end{aligned}$$

Then T is F-contraction on $\mathcal{H}(X)$.

Result 1 (continued)

Proof.

We demonstrate the claim for $N = 2$. Let $f_1, f_2 : X \to X$ be two F-contractions. Take $A, B \in \mathcal{H}(X)$ with $H(T(A), T(B)) > 0$. From Lemma's (iii), it follows that

$$\begin{aligned}
\tau + F\left(H(T(A), T(B))\right) &= \tau + F\left(H(f_1(A) \cup f_2(A), f_1(B) \cup f_2(B))\right) \\
&\leq \tau + F\left(\max\{H(f_1(A), f_1(B)), H(f_2(A), f_2(B))\}\right) \\
&\leq F\left(H(A, B)\right). \quad \square
\end{aligned}$$

□

Result 2

Theorem

Let (X, d) be a complete metric space and $\{f_n : n = 1, 2, ..., N\}$ a finite family of F-contractions on X. Define a mapping on $\mathcal{H}(X)$ as

$$\begin{aligned} T(A) &= f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) \\ &= \cup_{n=1}^{N} f_n(A), \text{ for each } A \in \mathcal{H}(X). \end{aligned}$$

Then

1. $T : \mathcal{H}(X) \to \mathcal{H}(X)$;
2. *T has a unique fixed point $U \in \mathcal{H}(X)$, that is* $U = T(U) = \cup_{n=1}^{k} f_n(U)$;
3. *for any initial set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), ...\}$ converges to a fixed point of T.*

Definition

Let (X, d) be a metric space. A mapping $T : \mathcal{H}(X) \rightarrow \mathcal{H}(X)$ is said to be a generalized F-contraction if there exists $F \in \digamma$ and $\tau > 0$ such that for any A, $B \in \mathcal{H}(X)$ with $H(T(A), T(B)) > 0$, the following holds:

$$\tau + F\left(H\left(T\left(A\right), T\left(B\right)\right)\right) \leq F(M_T(A, B)), \tag{1.7}$$

$$\begin{aligned} \text{where } M_T(A, B) \quad = \quad & \max\{H(A, B), H(A, T(A)), H(B, T(B)), \\ & \frac{H(A, T(B)) + H(B, T(A))}{2}, H(T^2(A), T(A)), \\ & H(T^2(A), B), H(T^2(A, TB)\}. \end{aligned}$$

The operator T defined above is also called generalized F-Hutchinson operator.

Definition IFS

Definition

Let X be a metric space. If $f_n : X \to X$, $n = 1, 2, ..., N$ are $F-$ contraction mappings, then $(X; f_1, f_2, ..., f_N)$ is called (or F-contractive) **generalzied iterated function system** (GIFS).

Thus generalized iterated function system consists of a metric space and finite family of $F-$ contraction mappings on X.

Definition

A nonempty compact set $A \subseteq X$ is said to be an **attractor** of the generalized F-contractive IFS T if

- $T(A) = A$ and
- there is an open set $V \subseteq X$ such that $A \subseteq V$ and $\lim_{k\to\infty} T^k(B) = A$ for any compact set $B \subseteq V$, where the limit is taken with respect to the Hausdorff metric.

Mair Result

Theorem

Let (X, d) *be a complete metric space and* $\{X : f_n, n = 1, 2, \cdots, N\}$ *a generalized iterated function system. Let* $T : \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by*

$$\begin{aligned} T(A) &= f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) \\ &= \cup_{n=1}^{N} f_n(A), \ \textit{for each } A \in \mathcal{H}(X). \end{aligned}$$

If T *is a generalized* $F-$ *Hutchinson operator, then* T *has a unique fixed point* $U \in \mathcal{H}(X)$, *that is*

$$U = T(U) = \cup_{n=1}^{k} f_n(U).$$

Moreover, for any initial set $A_0 \in \mathcal{H}(X)$, *the sequence of compact sets* $\{A_0, T(A_0), T^2(A_0), \ldots\}$ *converges to a fixed point of* T.

Fact

Fact

In above Theorem, if we take $\mathcal{S}(X)$ *the collection of all singleton subsets of* X, *then clearly* $\mathcal{S}(X) \subseteq \mathcal{H}(X)$. *Moreover, consider* $f_n = f$ *for each* n, *where* $f = f_1$ *then the mapping* T *becomes*

$$T(x) = f(x).$$

With this setting we obtain the following fixed point result.

Result

Corollary

Let (X,d) *be a complete metric space and* $\{X : f_n, n = 1, 2, \cdots, k\}$ *a generalized iterated function system. Let* $f : X \to X$ *be a mapping defined as in Remark. If there exists some* $F \in \digamma$ *and* $\tau > 0$ *such that for any* $x, y \in \mathcal{H}(X)$ *with* $d(f(x), f(y)) \neq 0$*, the following holds:*

$$\tau + F\left(d\left(fx, fy\right)\right) \leq F(M_f(x,y)),$$

where

$$\begin{aligned} M_T(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \\ & \quad d(f^2x,y), d(f^2x,fx), d(f^2x,fy)\}. \end{aligned}$$

Then f *has a unique fixed point* $x \in X$*. Moreover, for any initial set* $x_0 \in X$*, the sequence of compact sets* $\{x_0, fx_0, f^2x_0, ...\}$ *converges to a fixed point of* f*.*

Corollary

Corollary

Let (X, d) be a complete metric space and $(X; f_n, n = 1, 2, \cdots, k)$ be iterated function system where each f_i for $i = 1, 2, ..., k$ is a contraction self-mapping on X. Then $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}(X)$$

where each f_i for $i = 1, 2, ..., k$ is a contraction mapping on X, then T has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), ...\}$ converges to a fixed point of T.

Corollary

Corollary Let (X, d) be a complete metric space and $(X; f_n, n = 1, 2, \cdots, k)$ be IFS such that each f_i for $i = 1, 2, ..., k$ is a mapping on X satisfying

$$d(f_i x, f_i y)(d(f_i x, f_i y) + 1) \leq e^{-\tau} d(x, y)(d(x, y) + 1),$$
$$\text{for all } x, y \in X, \ f_i x \neq f_i y,$$

where $\tau > 0$. Then the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in main Theorem has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), ...\}$ converges to a fixed point of T.

Proof. By taking $F(\lambda) = \ln(\lambda^2 + \lambda) + \lambda$, $\lambda > 0$, we obtain that each mapping f_i for $i = 1, 2, ..., k$ on X satisfies

$$d(f_i x, f_i y)(d(f_i x, f_i y) + 1) \leq e^{-\tau} d(x, y)(d(x, y) + 1),$$
$$\text{for all } x, y \in X, \ f_i x \neq f_i y,$$

where $\tau > 0$.

Corollary (continued)

Thus the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}(X)$$

satisfies

$$H(T(A), T(B)) \leq \frac{1}{(1 + \tau\sqrt{H(A, B)})^2} H(A, B),$$

for all $A, B \in \mathcal{H}(X)$, $H(T(A), T(B)) \neq 0$. Using main Theorem, the result follows. $\square$

Example

Example

Let $X = [0,1] \times [0,1]$ and d be a Euclidean metric on X. Define $f_1, f_2 : X \to X$ as

$$f_1(x,y) = (\frac{1}{x+1}, \frac{y}{y+1}) \text{ and } f_2(x,y) = (\frac{\sin x}{\sin x + 1}, \frac{1}{\sin y + 1}).$$

Note that, for all $\mathbf{x} = (x_1, y_1), \mathbf{y} = (x_2, y_2) \in X$ with $\mathbf{x} \neq \mathbf{y}$,

$$\begin{aligned} d(f_1(\mathbf{x}), f_1(\mathbf{y})) &= d((\frac{1}{x_1+1}, \frac{y_1}{y_1+1}), (\frac{1}{x_2+1}, \frac{y_2}{y_2+1})) \\ &= \sqrt{\frac{(x_1-x_2)^2}{(x_1+1)^2(x_2+1)^2} + \frac{(y_1-y_2)^2}{(y_1+1)^2(y_2+1)^2}} \\ &< \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = d((x_1,y_1),(x_2,y_2)) = d(\mathbf{x},\mathbf{y}). \end{aligned}$$

Example (continued)

Also

$$d(f_2(\mathbf{x}), f_2(\mathbf{y})) = d((\frac{\sin x_1}{\sin x_1 + 1}, \frac{1}{\sin y_1 + 1}), (\frac{\sin x_2}{\sin x_2 + 1}, \frac{1}{\sin y_2 + 1}))$$

$$= \sqrt{\frac{(\sin x_1 - \sin x_2)^2}{(\sin x_1 + 1)^2(\sin x_2 + 1)^2} + \frac{(\sin y_1 - \sin y_2)^2}{(\sin y_1 + 1)^2(\sin y_2 + 1)^2}}$$

$$< \sqrt{(\sin x_1 - \sin x_2)^2 + (\sin y_1 - \sin y_2)^2}$$

$$\leq \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_1, y_1), (x_2, y_2)) = d(\mathbf{x}, \mathbf{y}).$$

Now there exists $\tau > 0$ such that

$$d(f_1(\mathbf{x}), f_1(\mathbf{y}))(1 + \tau\sqrt{d(\mathbf{x}, \mathbf{y})})^2 \leq d(\mathbf{x}, \mathbf{y}) \text{ and}$$

$$d(f_2(\mathbf{x}), f_2(\mathbf{y}))(1 + \tau\sqrt{d(\mathbf{x}, \mathbf{y})})^2 \leq d(\mathbf{x}, \mathbf{y})$$

are satisfied.

Example (continued)

Consider the iterated function system $\{\mathbb{R}^2; f_1, f_2\}$ with mapping $T : \mathcal{H}([0,1]^2) \to \mathcal{H}([0,1]^2)$ given as

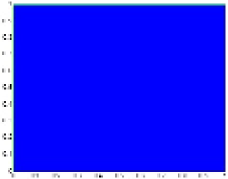
$$T(A) = f_1(A) \cup f_2(A) \text{ for all } A \in \mathcal{H}([0,1]^2).$$

For all $A, B \in \mathcal{H}([0,1]^2)$ with $H(T(A), T(B)) \neq 0$,

$$H(T(A), T(B))(1 + \tau\sqrt{H(A,B)})^2 \leq H(A,B)$$

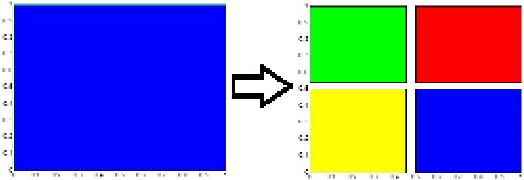
holds. Furthermore, we can analyze the convergence of T to the attractor of iterated function system in the Figure.

Example (continued)



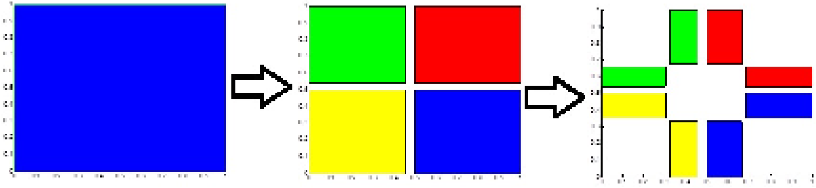
A

Example (continued)



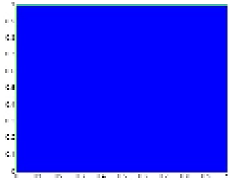
A T(A)

Example (continued)

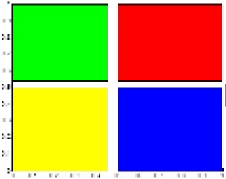


A $\quad$ T(A) $\quad$ $T^2(A)$

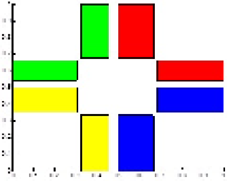
Example (continued)



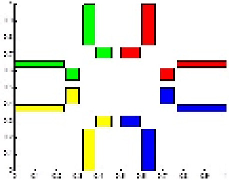
A



$T(A)$

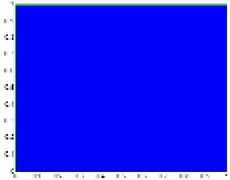


$T^2(A)$

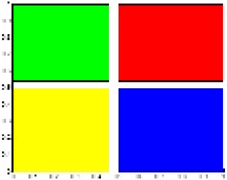


$T^3(A)$

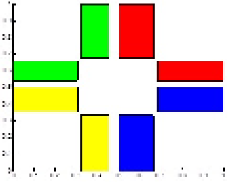
Example (continued)



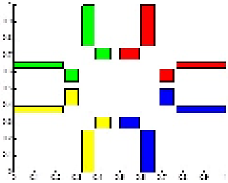
A



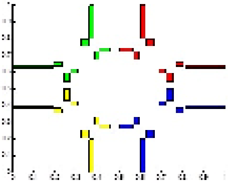
$T(A)$



$T^2(A)$

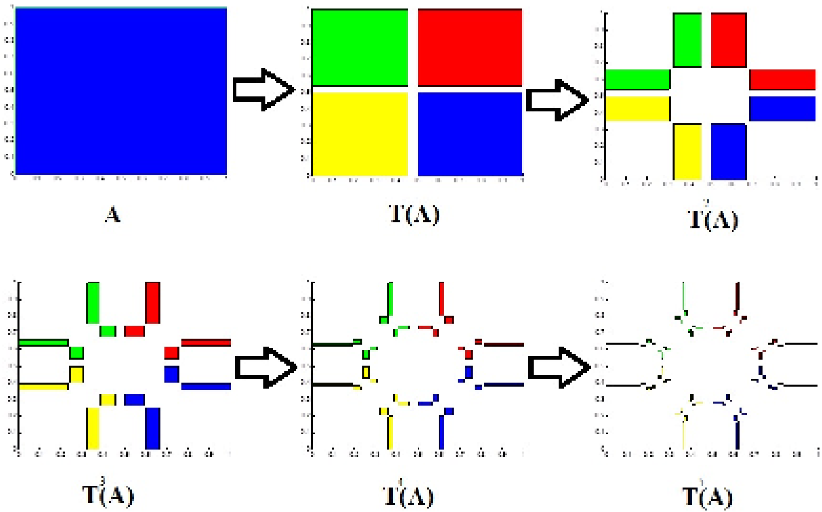


$T^3(A)$



$T^4(A)$

Example (continued)



References

- S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math., 1922, 3, 133-181.
- S. B. Nadler, Multivalued contraction mappings, Pacific J. Math., 1969, 30, 475-488.
- D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., 2012, 2012:94, 1-6.
- M. F. Barnsley, Fractals Everywhere, 2nd ed., Academic Press, San Diego, CA (1993).
- J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. Math., 1981, 30 (5), 713-747.

References 2

T. Nazir, S. Silvestrov and M. Abbas, Fractals of generalized F-Hutchinson operator, Waves, Wavelets and Fractals, 2016, 13, 222-246.

T. Nazir, S. Silvestrov and X. Qi, Fractals of generalized F-Hutchinson operator in b-metric spaces, Journal of Operators, Volume 2016, Article ID 5250394 (2016), 9 pages.

M. Abbas, M. R. Alfuraidan, T. Nazir, Common fixed points of multivalued F-contractions on metric spaces with a directed graph, Carpathian J. Math., 32 (2016), 1-12.

M. Abbas and T. Nazir, Attractor of the generalized contractive iterated function system, M. Ruzhansky, H. Dutta and R.P. Agarwal (Eds.) Mathematical Analysis and Applications: Selected Topics, Wiley, USA, ISBN: 9781119414421, 11 (2018), Pages 401-428.

Thanks