A parameter-uniform numerical method for singularly perturbed Burgers' equation

Justin M. Munyakazi

Joint work with Eshetu B. Derzie (ASTU) and Tekle G. Dinka (ASTU)

December 6, 2022

Outline of this talk

- Introducing the problem
- Some properties of the problem
- Numerical method
- Convergence analysis
- Numerical results

The problem

Consider the 1D singularly perturbed Burgers' IVP

$$L_\varepsilon u := \frac{\partial u}{\partial t}(x,t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0, \quad (1)$$

$$(x,t) \in D \equiv \Omega \times (0,T] \equiv (0,1) \times (0,T],$$

$$u(x,0) = f(x), \quad x \in \Omega, \quad (2)$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in [0,T], \quad (3)$$

$0 < \varepsilon \ll 1$: small parameter

The prescribed function $f(x)$ is sufficiently smooth.

Compatibility conditions $f(0) = 0 = f(1)$ at $(0,0)$ and $(1,0)$ of the domain $\bar{D}$.

Compatibility conditions + suitable continuity of $f(x)$: IBVP (1)-(3) has a unique solution $u(x,t)$.

When $\varepsilon \to 0$: boundary layer near to $x = 1$.

Properties of the continuous problem

Lemma

(Maximum principle) Let $\Phi(x,t) \in C^{2,1}(\bar{D})$*. Then if* $\Phi(x,t) \geq 0,\ \forall (x,t) \in \partial D$ *and* $L_\varepsilon \Phi(x,t) \geq 0, \forall (x,t) \in D$*, implies* $\Phi(x,t) \geq 0,\ \forall (x,t) \in \bar{D}$*.*

Proof.

Let (x^*,t^*) be such that $\Phi(x^*,t^*) = \min_{(x,t)\in\bar{D}} \Phi(x,t) < 0$. From the hypotheses we have $(x^*,t^*) \notin \partial D$, $\Phi_x(x^*,t^*) = \Phi_t(x^*,t^*) = 0$, and $\Phi_{xx}(x^*,t^*) > 0$. Thus

$$L_\varepsilon \Phi(x^*,t^*) = \Phi_t(x^*,t^*) - \varepsilon \Phi_{xx}(x^*,t^*) + \Phi(x^*,t^*)\Phi_x(x^*,t^*) = -\varepsilon \Phi_{xx}(x^*,t^*)$$

which contradicts to the assumption $L_\varepsilon \Phi \geq 0$ on D and hence $\Phi(x,t) \geq 0,\ \forall (x,t) \in \bar{D}$. □

Properties of the continuous problem (cont'd)

Lemma

Let $u(x,t)$ be the solution of (1)-(3). There exists a positive number C, which is independent of ε, such that for all small values of the perturbation parameter ε

$$|u(x,t) - f(x)| \leq Ct, \ \forall (x,t) \in \bar{D}.$$

Furthermore,

$$|u(x,t)| \leq C, \ \forall (x,t) \in \bar{D}.$$

Lemma

Let $u(x,t)$ be the solution of (1)-(3). By keeping x fixed along the line $\{(x,t) \in D : 0 \leq t \leq T\}$ the bounds of u_t and u_{tt} are given by

$$\frac{\partial^i u(x,t)}{\partial t^i} \leq C, \ \forall (x,t) \in D, \ \textit{for } i = 1, 2.$$

Numerical method

M, N: positive integers and $\bar{D}_M^N$: rectangular grid discretization of the rectangular domain $\bar{D} = \bar{\Omega} \times [0, T]$
Nodes are the (x_m, t_n) for $m = 0, 1, \cdots, M$ and $n = 0, 1, \cdots, N$

$$t_n = nk, \quad k = \frac{T}{N}, \quad n = 0, 1, \cdots, N,$$

$$x_m = mh, \quad h = \frac{1}{M}, \quad m = 0, 1, \cdots, M.$$

$u_m^n \approx u(x_m, t_n)$ at an arbitrary grid point (x_m, t_n) .

Time discretization with the averaged Crank-Nicolson method.
At the point $(x, t_{n+\frac{1}{2}})$, equation (1) can be approximated as

$$u_t^{n+\frac{1}{2}} = \varepsilon u_{xx}^{n+\frac{1}{2}} - (uu_x)^{n+\frac{1}{2}}, \tag{4}$$

where $u^{n+\frac{1}{2}}(x)$ is the approximate of $u(x, t_{n+\frac{1}{2}})$.
Using Taylor series expansion, the time derivative of u at $n + \frac{1}{2}$ time level is given by

$$u_t^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{k} + \tau, \tag{5}$$

where $\tau = -\frac{k^2}{24} u_{ttt}(x, \eta), \quad t_n \leq \eta \leq t_{n+1}$.
Substitute (5) in (4) and approximate the right side of equation (4) by the average of n and $n+1$ time level, we obtain

$$\frac{u^{n+1} - u^n}{k} + \tau = \varepsilon \frac{u_{xx}^{n+1} + u_{xx}^n}{2} - \frac{u^{n+1} u_x^{n+1} + u^n u_x^n}{2}. \tag{6}$$

Numerical method (cont'd)

$$-\varepsilon u_{xx}^{n+1} + u^{n+1}u_x^{n+1} + \frac{2}{k}u^{n+1} = \varepsilon(u_{xx}^n) - u^n u_x^n + \frac{2}{k}u^n - 2\tau. \qquad (7)$$

The semi-discrete problem can now be written as

$$u^0 = u(x,0) = f(x),$$

$$-\varepsilon u_{xx}^{n+1} + u^{n+1}u_x^{n+1} + \frac{2}{k}u^{n+1} = \varepsilon(u_{xx}^n) - u^n u_x^n + \frac{2}{k}u^n, \qquad (8)$$

with boundary conditions

$$u^{n+1}(0) = 0, \quad u^{n+1}(1) = 0, \quad n = 0, 1, \cdots, N-1.$$

Equation (8) is a nonlinear two-point boundary value problem for each $n = 0, 1, \cdots, N-1$.

Numerical method (cont'd)

Linearize the nonlinear BVP (8) by using Newton's quasilinearization approach. The nonlinear eq. (8) can be written as

$$\varepsilon u_{xx}^{n+1} = F\left(u^{n+1}, u_x^{n+1}\right), \tag{9}$$

where

$$F\left(u^{n+1}, u_x^{n+1}\right) = u^{n+1}u_x^{n+1} + \frac{2}{k}u^{n+1} - \varepsilon(u_{xx}^n) + u^n u_x^n - \frac{2}{k}u^n. \tag{10}$$

Suppose $\left(u^{n+1,(j)}(x), u_x^{n+1,(j)}(x)\right)$ be the (j)th nominal solution to (9). Taking the Taylor series expansion of F up to first-order terms around the nominal solution $\left(u^{n+1,(j)}, u_x^{n+1,(j)}\right)$, we obtain

$$F\left(u^{n+1,(j+1)}, u_x^{n+1,(j+1)}\right) = F\left(u^{n+1,(j)}, u_x^{n+1,(j)}\right) + \left(u^{n+1,(j+1)} - u^{n+1,(j)}\right) F_{u^{n+1}}\left(u^{n+1,(j)}, u_x^{n+1,(j)}\right) + \left(u_x^{n+1,(j+1)} - u_x^{n+1,(j)}\right) F_{u_x^{n+1}}\left(u^{n+1,(j)}, u_x^{n+1,(j)}\right)$$

Substitute (11) in (9) and after simplification, we obtain the following sequence of linear second order differential equations

$$-\varepsilon u_{xx}^{n+1,(j+1)} + u^{n+1,(j)} u_x^{n+1,(j+1)} + \left(u_x^{n+1,(j)} + \frac{2}{k}\right) u^{n+1,(j+1)} = \varepsilon u_{xx}^{n,(j+1)} + \left(\frac{2}{k} - u_x^{n,(j+1)}\right) u^{n,(j+1)} + u^{n+1,(j)} u_x^{n+1,(j)}, \quad (11)$$

with BCs

$$u^{n+1,(j+1)}(0) = 0, \quad u^{n+1,(j+1)}(1) = 0, \quad n = 0, 1, \cdots, N-1,$$

Iteration index $j = 0, 1, 2, \cdots$, init. guess: $u^{n+1,(0)}(x)$.

For the sake of simplicity, use the following notations

$$U(x) = u^{n+1,(j+1)}(x), \quad a(x) = u^{n+1,(j)}(x), \quad b(x) = u_x^{n+1,(j)}(x) + \frac{2}{k},$$

$$H(x) = \varepsilon u_{xx}^{n,(j+1)} + \left(\frac{2}{k} - u_x^{n,(j+1)}\right) u^{n,(j+1)} + u^{n+1,(j)} u_x^{n+1,(j)}.$$

Eq. (11) can be written as

$$L_\varepsilon^N U(x) = H(x), \ x \in \Omega, \tag{12}$$

with BCs $U(0) = 0, \quad U(1) = 0,$

$$L_\varepsilon^N U(x) = -\varepsilon U_{xx} + a(x) U_x + b(x) U.$$

Assume $a(x), b(x)$ and $H(x)$ are sufficiently smooth functions and $a(x) \geq \alpha > 0, \ b(x) \geq \beta > 0$.
Under these conditions, the linear SP BVP (12) has a unique solution that exhibits a boundary layer at $x = 1$ when $\varepsilon \to 0$.

Numerical method (cont'd)

Let $\left(u^{n+1,(j)}(x), (u_x)^{n+1,(j)}(x)\right)$ be the jth nominal solution to (9) and

$$p = \max_{\|u^{n+1}\| \leq 1} |F_{u^{n+1}}|, \;\; q = \max_{\|u^{n+1}\| \leq 1} |F_{u^{n+1}u^{n+1}}|. \tag{13}$$

In [2] the following inequality was proved

$$\max_x |u^{n+1,(j+2)} - u^{n+1,(j+1)}| \leq \frac{q}{8\varepsilon(1-p/4\varepsilon)} \max_x |u^{n+1,(j+1)} - u^{n+1,(j)}|^2 \tag{14}$$

Thus the sequence of linear equations converges quadratically provided that

$$\frac{q}{8\varepsilon(1-p/4\varepsilon)} < 1. \tag{15}$$

The inequality (15) holds by choosing an appropriate initial approximation $u^{n+1,(0)}$ such that $|u^{n+1,(1)} - u^{n+1,(0)}|$ is small.

Numerical method (cont'd)

The linear L_ε^N defined in (12) satisfies the following maximum principle.

Lemma

(Semi-discrete maximum principle) Assume that $\phi(0) \geq 0,\ \phi(1) \geq 0$ *and* $L_\varepsilon^N \phi(x) \geq 0$ *for all* $x \in \Omega$ *then* $\phi(x) \geq 0$ *for all* $x \in \bar{\Omega}$.

Proof.

Suppose there exist x^* such that $\phi(x^*) = \min_{x \in \bar{\Omega}} \phi(x) < 0$. From the hypothesis we have $x^* \neq 0,\ x^* \neq 1,\ \phi_x(x^*) = 0$ and $\phi_{xx}(x^*) > 0$. Then, we have

$$\begin{aligned} L_\varepsilon^N \phi(x^*) &= -\varepsilon \phi_{xx}(x^*) + a(x^*)\phi_x(x^*) + b(x^*)\phi(x^*) \\ &= -\varepsilon \phi_{xx}(x^*) + b(x^*)\phi(x^*) < 0, \text{ since } b(x^*) > 0, \end{aligned}$$

which contradicts the assumption and hence $\phi(x) \geq 0$ for all $x \in \bar{\Omega}$. □

Numerical method (cont'd)

Local error $e_{n+1} = u(x, t_{n+1}) - U(x)$.

Lemma

The local error estimate in the temporal direction is given by

$$\|e_{n+1}\|_\infty \leq Ck^3.$$

The global error of the time semi-discretization is $E_n = \sum_{i=1}^{n} e_i$.

Theorem

The global error estimate at t_n is given by

$$\|E_n\|_\infty \leq Ck^2, \quad nk \leq T.$$

Numerical method (cont'd)

Lemma

Let $U(x)$ be the solution of (12), the bounds of $U(x)$ and its derivatives are

$$\left|\frac{\partial^i U}{\partial x^i}\right| \leq C\left(1+\varepsilon^{-i}\exp(-\alpha(1-x)/\varepsilon)\right), \quad i=0,1,2,3. \tag{16}$$

Numerical method (cont'd)

$$L_\varepsilon^{M,N} U_m := -\varepsilon \left[\frac{U_{m+1} - 2U_m + U_{m-1}}{\psi_m^2}\right] + a_m \left[\frac{U_m - U_{m-1}}{h}\right] + b_m U_m = H_m, \tag{17}$$

where $\psi_m^2 = \dfrac{h\varepsilon}{a_m}\left(\exp(\dfrac{a_m}{\varepsilon}) - 1\right), \quad m = 1, 2, \cdots, M-1.$

$$E_m U_{m-1} + F_m U_m + G_m U_{m+1} = H_m, \quad m = 1, 2, \cdots, M-1, \tag{18}$$

where

$$E_m = \frac{-\varepsilon}{\psi_m^2} - \frac{a_m}{h}, \tag{19}$$

$$F_m = \frac{2\varepsilon}{\psi_m^2} + \frac{a_m}{h} + b_m, \tag{20}$$

$$G_m = -\frac{\varepsilon}{\psi_m^2}, \tag{21}$$

$$H_m = \frac{\varepsilon}{\psi_m^2}(u_{m-1}^{n,(j+1)} - 2u_m^{n,(j+1)} + u_{m+1}^{n,(j+1)}) \tag{22}$$

Convergence analysis

Lemma

(Fully discrete maximum principle) Suppose for any mesh function ϕ^M, $\phi^M(x_0) \geq 0$, $\phi^M(x_M) \geq 0$ and $L_\varepsilon^{M,N} \phi_m^M \geq 0$ for $1 \leq m \leq M-1$ then $\phi_m^M \geq 0$ for $0 \leq m \leq M$.

Proof.

Suppose there exist m^* be such that $\phi^M(x_m^*) = \min\limits_{0 \leq m \leq M} \phi^M(x_m) < 0$. From the hypothesis we have $m^* \neq 0$, $m^* \neq M$. Thus, we have

$$
\begin{aligned}
L_\varepsilon^{M,N} \phi^M(x_{m^*}) &= -\varepsilon \left[\frac{\phi_{m^*+1}^M - 2\phi_{m^*}^M + \phi_{m^*-1}^M}{\psi_{m^*}^2} \right] + a(x_{m^*}) \left[\frac{\phi_{m^*}^M - \phi_{m^*-1}^M}{h} \right] \\
&\quad + b(x_{m^*}) \phi_{m^*}^M < 0,
\end{aligned}
$$

which contradicts the assumption and hence $\phi_m^M \geq 0$ for $0 \leq m \leq M$. □

Convergence analysis (cont'd)

Lemma

Let ϕ^M be any mesh function such that $\phi_0^M = \phi_M^M = 0$, then

$$|\phi_m^M| \leq \frac{1}{\beta} \max_{1\leq i\leq M-1} |L_\varepsilon^{M,N}\phi_i^M|, \quad 0 \leq m \leq M.$$

Proof.

Consider the barrier functions $\Psi_m^{\pm} = \dfrac{1}{\beta} \max_{1\leq i\leq M-1} |L_\varepsilon^{M,N}\phi_i^M| \pm \phi_m^M$. We see that $\Psi^{\pm}(0) =\geq 0, \Psi^{\pm}(1) \geq 0$. and $L_\varepsilon^{M,N}\Psi_m^{\pm} \geq 0, \;\; 1 \leq m \leq M-1$ An application of Lemma 8 yields
$|\phi_m^M| \leq \frac{1}{\beta} \max_{1\leq i\leq M-1} |L_\varepsilon^{M,N}\phi_i^M|, \quad 0 \leq m \leq M.$ □

Convergence analysis (cont'd)

Theorem

(Error in the spatial direction) Let $U(x)$ be the solution of the problem (12) after temporal discretization and U_m be the solution of (17) after the full discretization. Then, the error estimate is given by

$$|U(x_m) - U_m| \leq CM^{-1},\ 0 \leq m \leq M.$$

Proof.

The local trunc. error of the FOFDM(17) is given by

$$\begin{aligned} L_\varepsilon^{M,N}(U(x_m) - U_m) =& [-\varepsilon U''(x_m) + a(x_m)U'(x_m) + b(x_m)U(x_m)] \\ & -[-\varepsilon \frac{h^2}{\psi^2} D^+D^-U(x_m) + a(x_m)D^-U(x_m) + b(x_m)U(x_m)] \\ =& \varepsilon(\frac{h^2}{\psi^2} D^+D^- - \frac{d^2}{dx^2})U(x_m) + a(x_m)(\frac{d}{dx} - D^-)U(x_m), \end{aligned}$$

where $D^+D^-U(x_m) = \frac{U(x_{m+1}) - 2U(x_m) + U(x_{m-1})}{h^2}$.

Convergence analysis (cont'd)

Proof.

$$|L_\varepsilon^{M,N}(U(x_m)-U_m)| \leq \varepsilon|(\frac{h^2}{\psi^2}-1)D^+D^-U(x_m)| + \varepsilon|(D^+D^- - \frac{d^2}{dx^2})U(x_m)|$$
$$+|a(x_m)(\frac{d}{dx}-D^-)U(x_m)|. \quad (23)$$

Simplifying as,

$$\frac{h^2}{\psi^2}-1 = -\frac{a(\eta)h}{2\varepsilon+ha(\eta)}, \quad (24)$$
$$(D^+D^- - \frac{d^2}{dx^2})U(x_m) = \frac{h^2}{16}\frac{d^4U}{dx^4}(\eta), \quad (25)$$
$$(\frac{d}{dx}-D^-)U(x_m) = \frac{h}{2}\frac{d^2U}{dx^2}(\eta), \quad (26)$$

for some η, $x_{m-1} \leq \eta \leq x_{m+1}$.

Proof.

Using (24)-(26), equation (23) leads to,

$$|L_\varepsilon^{M,N}(U(x_m) - U_m)| \leq \varepsilon \frac{a(\eta)h}{2\varepsilon + ha(\eta)} \| \frac{d^2 U}{dx^2}(\eta) \| + \varepsilon \frac{h^2}{16} \| \frac{d^4 U}{dx^4}(\eta) \| + a(x_m) \frac{h}{2} \| \frac{d}{d} \tag{27}$$

Using the bounds of derivatives in (16) and the boundedness of $a(x)$, we obtain

$$\begin{aligned} |L_\varepsilon^{M,N}(U(x_m) - U_m)| \leq & \frac{h}{2\varepsilon + ha(\eta)} C\varepsilon \left(1 + \varepsilon^{-2} \exp\left(\frac{-\alpha(1 - x_m)}{\varepsilon}\right)\right) + Ch^2\varepsilon \\ & + Ch \left(1 + \varepsilon^{-2} \exp\left(\frac{-\alpha(1 - x_m)}{\varepsilon}\right)\right) \leq \\ & + Ch \left(1 + \varepsilon^{-2} \exp\left(\frac{-\alpha(1 - x)}{\varepsilon}\right)\right) \end{aligned}$$

Convergence analysis (cont'd)

Proof.

Apply limit as ε approaches to zero, we have

$$\lim_{\varepsilon\to 0} |L_\varepsilon^{M,N}(U(x_m) - U_m)| \leq Ch, \quad m = 0, 1, \cdots, M.$$

Lemma 9 leads to

$$\lim_{\varepsilon\to 0} |U(x_m) - U_m| \leq Ch. \tag{28}$$

□

Convergence analysis (cont'd)

The combination of Theorems 6 and 10 gives the following ε-uniform convergence estimate for the total discrete scheme.

Theorem

Let $u(x,t)$ be the solution of the problem (1) and U_m be the approximation to the solution $u(x_m, t_{n+1})$ of the fully discretized scheme given by (17). Then, the error estimate for the totally discrete scheme is given by

$$|u(x_m, t_{n+1}) - U_m| \leq C(h + k^2), \ 0 \leq m \leq M.$$

Convergence analysis (cont'd)

The maximum nodal error

$$E_\varepsilon^{M,N} = \max_{(x_m,t_n)\in D_M^N} |u_{m;M}^{n;N} - u_{m;2M}^{n;2N}|,$$

The ε-uniform maximum nodal error is defined by $E^{M,N} = \max_\varepsilon E_\varepsilon^{M,N}$.

Rate of convergence $R_\varepsilon^{M,N} = \log_2\left(\frac{E_\varepsilon^{M,N}}{E_\varepsilon^{2M,2N}}\right)$, and the numerical ε-uniform rate of convergence is defined as $R^{M,N} = \max_\varepsilon R_\varepsilon^{M,N}$.

Stopping criterion for Newton quasilinearization:

$$\|u_m^{n+1,(j+1)} - u_m^{n+1,(j)}\| \leq 10^{-6}. \tag{29}$$

Numerical results

Example

Consider the singularly perturbed Burgers' problem:

$$\frac{\partial u}{\partial t}(x,t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + u\frac{\partial u}{\partial x}(x,t) = 0, \quad (x,t) \in (0,1) \times (0,1],$$

subject to the conditions:

$$u(x,0) = \sin(\pi x),\ 0 < x < 1,$$
$$u(0,t) = 0 = u(1,t), 0 \le t \le 1.$$

The Fourier series solution for this example is known using Cole-Hopf transformation [?], that is,

$$u(x,t) = 2\varepsilon\pi \frac{\sum_{n=1}^{\infty} \exp(-\varepsilon n^2 \pi^2 t) n A_n \sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} \exp(-\varepsilon n^2 \pi^2 t) A_n \cos(n\pi x)}, \quad (30)$$

Numerical results (cont'd)

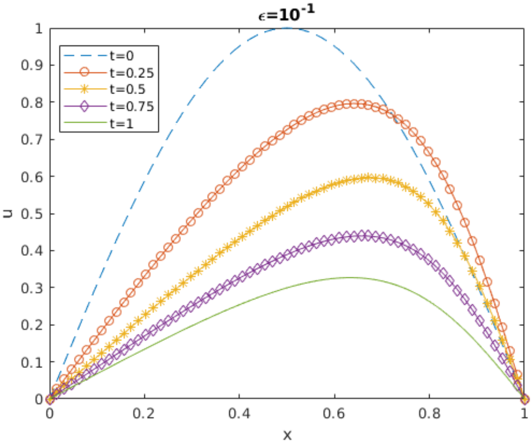

Figure: Numerical solutions for Example 12 at different time levels with $M = 64,\ N = 40$.

Numerical results (cont'd)

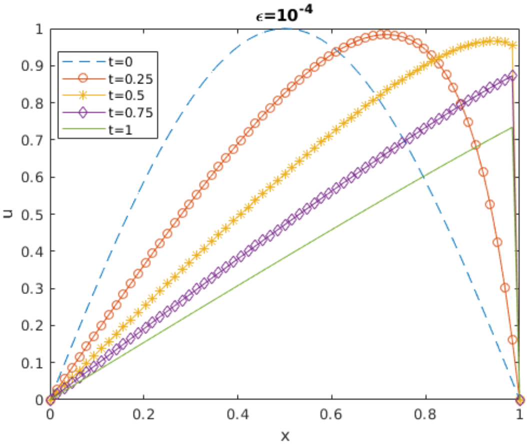

Figure: Numerical solutions for Example 12 at different time levels with $M = 64,\ N = 40$.

Numerical results (cont'd)

$\varepsilon \downarrow$	N=20	N=40	N=80	N=160	320
10^0	1.041e-02	2.294e-03	5.120e-04	1.266e-04	3.150e-05
	2.182	2.164	2.016	2.006	
10^{-2}	8.208e-03	2.260e-03	5.752e-04	1.439e-04	3.599e-05
	1.861	1.974	1.998	2.000	
10^{-4}	1.354e-02	4.330e-03	1.167e-03	2.921e-04	7.305e-05
	1.645	1.892	1.998	2.000	
10^{-6}	1.354e-02	4.330e-03	1.167e-03	2.921e-04	7.305e-05
	1.645	1.892	1.998	2.000	
10^{-8}	1.354e-02	4.330e-03	1.167e-03	2.921e-04	7.305e-05
	1.645	1.892	1.998	2.000	
10^{-10}	1.354e-02	4.330e-03	1.167e-03	2.921e-04	7.305e-05
	1.645	1.892	1.998	2.000	
10^{-12}	1.354e-02	4.330e-03	1.167e-03	2.921e-04	7.305e-05
	1.645	1.892	1.998	2.000	

Table: Maximum nodal errors and numerical rate of convergence for Example 12 at the number of intervals N by fixing the spatial mesh size $M = 64$.

Numerical results (cont'd)

$\varepsilon \downarrow$	M=64	M=128	M=256	M=512	1024
10^{0}	6.355e-05	1.589e-05	3.972e-06	9.931e-07	2.483e-07
	2.000	2.000	2.000	2.000	
10^{-2}	2.408e-02	5.413e-03	1.310e-03	3.274e-04	8.168e-05
	2.153	2.047	2.000	2.003	
10^{-4}	5.802e-02	3.501e-02	1.927e-02	1.003e-02	4.835e-03
	0.729	0.862	0.942	1.052	
10^{-6}	5.802e-02	3.500e-02	1.921e-02	1.002e-02	5.100e-03
	0.729	0.866	0.939	0.974	
10^{-8}	5.802e-02	3.500e-02	1.921e-02	1.002e-02	5.100e-03
	0.729	0.866	0.939	0.974	
10^{-10}	5.802e-02	3.500e-02	1.921e-02	1.002e-02	5.100e-03
	0.729	0.866	0.939	0.974	
10^{-12}	5.802e-02	3.500e-02	1.921e-02	1.002e-02	5.100e-03
	0.729	0.866	0.939	0.974	

Table: Maximum nodal errors and numerical rate of convergence for Example 12 at the number of intervals M by fixing the temporal mesh size $N = 10$.

References

Eshetu B. Derzie, Justin B. Munyakazi, Tekle Gemechu, A parameter-uniform numerical method for singularly perturbed Burgers' equation, *Computational and Applied Mathematics* (2022) 41:247.

A. Kaushik, M. D. Sharma, A uniformly convergent numerical method on non-uniform mesh for singularly perturbed unsteady Burger-Huxley equation, *Applied Mathematics and Computation* **195** (2008)688-706.