JC-spaces and *J*-frames

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SAMS2022

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JConnected Spaces

- A space X is JC-**space** if whenever X is a union of two of its *closed* subsets with a connected intersection, then one of the closed sets is connected.

Theorem

Every connected space is a JC-space.

Proof.

Idea: If we express a connected space X as a union of two of its closed subsets whose intersection is connected, then both of the two closed subsets are connected. □

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Theorem

If a space X is a union of two non-empty disjoint connected subsets, then it is a JC-space.

EXAMPLES

1. Consider $X = (-\infty, 0) \cup (0, \infty)$ as a subspace of $\mathbb{R}$. Then X is a disconnected JC-space.

2. **GJ**: $\mathrm{cl}\,\mathbb{R}^+ \setminus \mathbb{R}^+$ and $\mathrm{cl}\,\mathbb{R}^- \setminus \mathbb{R}^-$ are disjoint connected subsets of $\beta\mathbb{R}$ and their union is $\beta\mathbb{R} \setminus \mathbb{R}$.

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Non-examples

Non-examples of JC-sapces

- 1. Consider the space $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Put

$$A = (-\infty, 0] \cap \mathbb{Q} \quad \text{and} \quad B = [0, \infty) \cap \mathbb{Q}.$$

Then $\mathbb{Q} = A \cup B$, A, B are closed subsets in $\mathbb{Q}$ and $A \cap B = \{0\}$ is connected. However neither A nor B is connected in $\mathbb{Q}$. Thus $\mathbb{Q}$ is not a JC-space.

- 2. Consider $X = (-3, 0) \cup (0, 1) \cup (1, 2)$, as subspace of $\mathbb{R}$. Then X is not a JC-space: Simply note that

$$A = (-3, 0) \cup (0, 1) \quad \text{and} \quad B = (0, 1) \cup (1, 2)$$

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More on JC-spaces

Basic Results

- 1. Let $f : X \to Y$ be a monotone closed mapping from X onto Y. Then X is a JC-space if and only if Y is a JC-space.
- 2. Let Z be a connected space and Y be any space. Then $Z \times Y$ is a JC-space if and only if Y is a JC-space.
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F-compactness

J-spaces

- 1. A space X is a *J-space* if any closed binary cover with a compact intersection has a compact member (E. Michael, 2000).
- 2. Let L be a complete lattice. An element $a \in L$ is called *F-compact* if whenever $a \wedge (\bigwedge S) = 0$ for some $S \subseteq L$, then there exists a finite $F \subseteq S$ such that $a \wedge (\bigwedge F) = 0$.
- 3. Let $CL(X)$ be the lattice of closed subsets of X.

Theorem

$A \in CL(X)$ is F-compact if and only if A is a compact subset of X.

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J-frames

- 1. We say that a frame L is a *J-frame* if whenever $a \wedge b = 0$ in L and the frame $\mathfrak{c}(a \vee b)$ is compact, then $\mathfrak{c}(a)$ or $\mathfrak{c}(b)$ is compact.
- 2. A Hausdorff space X is a J-space if and only if $\mathfrak{O}X$ is a J-frame.

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A) $\mathfrak{O}\mathbb{R}$ is not a J-frame.

B) $\mathfrak{O}\mathbb{R}^{+}$ is a J-frame.

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J-frames with no points

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A frame with no points is a J-frame if and only if it is connected.

Example

Let L be a Boolean frame which has no points. Such an L is disconnected non-spatial frame which is not J-frame.

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The following conditions are equivalent for a frame L:

1. L is a J-frame.

2. Whenever $a \in L$, and $\mathfrak{c}(a \vee a^*)$ compact, then either $\mathfrak{c}(a)$ or $\mathfrak{c}(a^*)$ is compact.

Proof.

(1) $\Longrightarrow$ (2) This is clear from the definition of J-frame.

(2) $\Longrightarrow$ (1) Suppose $\mathfrak{c}(a \vee b)$ is compact, with $a \wedge b = 0$ in L. We need to show that $\mathfrak{c}(a)$ or $\mathfrak{c}(b)$ is compact.Now, $b \leq a^*$ implies that $(a \vee b) \leq (a \vee a^*)$, therefore $\mathfrak{c}(a \vee a^*)$ is compact.Thus, $\mathfrak{c}(a)$ or $\mathfrak{c}(a^*)$ is compact, by hypothesis.If $\mathfrak{c}(a)$ is compact then we are done.If $\mathfrak{c}(a^*)$ is compact, then $\mathfrak{c}(a^*) \vee \mathfrak{c}(a \vee b)$ is compact. But $\mathfrak{c}(a^*) \vee \mathfrak{c}(a \vee b) = \mathfrak{c}(a^* \wedge (a \vee b)) = \mathfrak{c}(a^* \wedge b)$.So $\mathfrak{c}(a^* \wedge b)$ is compact, and hence $\mathfrak{c}(b)$ is compact. □

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(2) $\implies$ (1) Suppose $\mathfrak{c}(a \vee b)$ is compact, with $a \wedge b = 0$ in L. We need to show that $\mathfrak{c}(a)$ or $\mathfrak{c}(b)$ is compact.Now, $b \leq a^*$ implies that $(a \vee b) \leq (a \vee a^*)$, therefore $\mathfrak{c}(a \vee a^*)$ is compact.Thus, $\mathfrak{c}(a)$ or $\mathfrak{c}(a^*)$ is compact, by hypothesis.If $\mathfrak{c}(a)$ is compact then we are done.If $\mathfrak{c}(a^*)$ is compact, then $\mathfrak{c}(a^*) \vee \mathfrak{c}(a \vee b)$ is compact. But $\mathfrak{c}(a^*) \vee \mathfrak{c}(a \vee b) = \mathfrak{c}(a^* \wedge (a \vee b)) = \mathfrak{c}(a^* \wedge b)$.So $\mathfrak{c}(a^* \wedge b)$ is compact, and hence $\mathfrak{c}(b)$ is compact. □

Characterisation

Theorem

The following conditions are equivalent for a frame L:
1. L is a J-frame.
2. Whenever $a \in L$, and $\mathfrak{c}(a \vee a^*)$ compact, then either $\mathfrak{c}(a)$ or $\mathfrak{c}(a^*)$ is compact.

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Relative connectedness and J-frames

DEFINITION:

A sublocale S of a frame L is *relatively connected* in L if whenever $S \subseteq U$, where U is an open sublocale of L with $U = U_1 \vee U_2$ for some open sublocales U_1, U_2 of L such that $U_1 \cap U_2 = \mathbb{O}$, then $S \cap U_1 = \mathbb{O}$ or $S \cap U_2 = \mathbb{O}$.

Theorem

The following conditions are equivalent for a non-compact completely regular frame L:

- *1. L is a J-frame.*
- *2. $M \smallsetminus L$ is relatively connected in M, for any compactification M of L.*
- *3. $\beta L \smallsetminus L$ is relatively connected in βL.*
- *4. If $\beta L = c(I) \vee c(J)$ for some $I, J \in \beta L$ and $c(I) \cap c(J) \subseteq L$, then $c(I) \subseteq L$ or $c(J) \subseteq L$.*

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