Comparative performance of time spectral methods for solving hyperchaotic finance and cryptocurrency systems

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December 6, 2022

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Introduction & Literature

It is without a doubt that an analytical solution for the nonlinear hyperchaotic finance system is almost unachievable. For this reason, we shall rely on numerical methods. In the of field numerical methods for solving differential equations, two main classes can be distinguished:

- Classical methods
- Spectral methods

Introduction & Literature

- For more on spectral methods, see Shen et al. (2011); Fornberg and Driscoll (1999)
- Babolian Babolian and Hosseini (2002) introduced a modified spectral method that is more efficient than the normal spectral method. Various modified and quadrature rules can be found in the literature of spectral methods, including quadrature based on Chebyshev polynomials.
- Bhrawy Bhrawy and Alofi (2013) introduces an operational matrix to the shifted Chebyshev method to generate an even faster algorithm for fractional integration.

- Driscoll Driscoll (2010) presents a fast algorithm based on operational matrices in which the matrices have a lower density.
- The Chebfun package of Matlab Trefethen (2000) is used in the algorithm, as it exploits results from approximation theory, spectral methods, and object-oriented software design.
- Matlab Differentiation Matrix Suite (DMS) package Weideman and Reddy (2000)
- Trif Trif (2011) introduces the chebpack package that is based on the Chebyshev-Tau method where the focus is more on the spectral space of coefficients rather than the physical space.

Chebyshev polynomials

The Chebyshev polynomial $T_n(x)$ of 1st kind is a polynomial in $x \in [-1, 1]$ of degree $n > 0$ defined by the relation:

$$\begin{aligned} T_n(x) &= \cos n\theta, \text{ for } x = \cos\theta \\ ie. \quad T_n(x) &= \cos(n \arccos(x)) \end{aligned}$$

From the trigonometric relation.

$$\cos(n\theta) + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta \tag{1}$$

we get

$$T_0(x) = 1, \tag{2}$$

$$T_1(x) = x, \tag{3}$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots \tag{4}$$

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which in turn can be expressed in a matrix form as:

$$\begin{bmatrix} 1 & & & & \\ -2x & 1 & & & \\ 1 & -2x & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2x & 1 \end{bmatrix} \begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ -x \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (5)$$

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The zeros of T_n are the points

$$x_k = -\cos \frac{(k-\frac{1}{2})\pi}{n}, \qquad k = 1, 2, \ldots, n. \tag{6}$$

The set $\{x_k\}_k$ is termed as collocation points, also called Chebyshev points of first kind. For any point x, the set $\{T_0(x), T_1(x), \ldots\}$ is an orthogonal basis according to the weighted inner product defined by:

$$< f, g >= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx \tag{7}$$

for any continuous function f, g defined on $[-1, 1]$. This means that for any polynomial of degree $n > 0$, there exists a unique set of coefficients $\{c_1, c_2, ..., c_n\}$ such that

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x). \tag{8}$$

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Considering that polynomials are dense in $C([-1,1])$,

Theorem

Let f be a Lipschitz continuous function on the interval [-1,1]. Then f admits a unique representation as a series of the form:

$$u(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k T_k(x). \tag{9}$$

where $T_k(x)$ are Chebyshev polynomials,

$$c_k = \frac{2}{\pi} \int_{-1}^{1} \frac{u(x)\, T_k(x)}{\sqrt{1-x^2}} dx, \qquad k = 0, 1, 2, 3, \ldots \tag{10}$$

This series converges uniformly and absolutely.

A Chebyshev approximation of order $n > 0$ of a function u continuous on an interval $[-1, 1]$ is defined by

$$u_n(x) = \sum_{k=0}^{n} c_k T_k(x) \quad (11)$$

$$= \underline{c} \cdot T(x) \quad (12)$$

where $\underline{c} = (c_0, c_1, \ldots, c_n)$ is the coefficient vector associated with the approximation u_n. It is usually termed as the spectral representation of u_n. The set of Chebyshev coefficient vectors $\{\underline{c}\}$ of continuous functions on $[-1, 1]$ is referred to as the **frequency space**.

This means u can be represented by a vector $\underline{v} = (u(x_0), u(x_1), ..., u(x_n))$. We shall call $\underline{v}$ the **physical representation** of u.

On the collocation point, one writes

$$\underline{v}(x) = T(x).\underline{c}, \quad (13)$$

$$(v(x_0), ..., v(x_n)) = \left(\sum_{k=0}^{n} c_k T_k(x_0), ..., \sum_{k=0}^{n} c_k T_k(x_n), \right) \quad (14)$$

where T is the matrix defined as follows

$$T = \begin{bmatrix} T_0(x_0) & T_1(x_0) & \dots & T_n(x_0) \\ T_0(x_1) & T_1(x_1) & \dots & T_n(x_1) \\ \vdots & & \ddots & \vdots \\ T_0(x_n) & T_1(x_n) & \dots & T_n(x_n) \end{bmatrix}.$$

$$\text{Since} \quad \underline{v} = T\underline{c} \quad \Longrightarrow \quad \underline{c} = T^{-1}\underline{v},$$

From the nature of $T_k's$, The matrix T is sparse and FFT enables to get T^{-1}.

Some useful properties

Consider two functions a and u of a variable x, then the product $a(x) \cdot u(x)$ admits also a spectral representation, denoted as $\underline{\phi}$ which is defined by

$$\underline{\phi} = \mathbf{a}.\underline{c} \tag{15}$$

where $\mathbf{a}$ is termed as the matrix representation of the function $a(x)$, see Driscoll (2010) and $\underline{c}$ is defined as in (12)
An efficient way of getting matrix $\mathbf{a}$ is to write the product in its discrete form.

$$\text{Since} \quad a(x)f(x) = \left[\sum_{k=0}^{n} a_k T_k(x)\right]\left[\sum_{k=0}^{n} c_k T_k(x)\right], \tag{16}$$

$$\text{then} \quad \sum_{k=0}^{n} \phi_k T_k(x) = \sum_{k=0}^{n}\sum_{l=0}^{n} \alpha_{kl} a_k c_l T_k T_l \tag{17}$$

for some coefficients α_{kl}, $0 \leq k, l \leq n$.

In addition, given the following relation

$$T_k(x)\,T_l(x) = \frac{1}{2}\left[T_{k+l}(x) + T_{|k-l|}(x)\right], \qquad \text{for all } k, l = 0, 1, \ldots n \tag{18}$$

and in rearranging terms properly, it brings to existence a matrix **a** such that

$$\sum_{k=0}^{n} \phi_k T_k(x) = \sum_{k=0}^{n}\left[\sum_{l=0}^{n} \mathbf{a}_{kl} c_l\right] T_k(x).$$

In the frequency space, this will written in the form

$$\underline{\phi} = \mathbf{a}\underline{c}. \tag{19}$$

Differentiation and integration

In view of equation (11) and by differentiation, $u'(x)$ is given by

$$u'(x) = \sum_{k=0}^{n} c_k T_k'(x). \tag{20}$$

The differentiation of relation (4) and (3) implies

$$T_0 = T_1', \qquad T_1 = \frac{T_2'}{2}, \tag{21}$$

$$T_{n+1}(x) = nT_{n-1}'(x) - 2(1-x^2)T_n'(x) \tag{22}$$

$$\text{ie.} \quad T_n = \frac{T_{n+1}'}{2(n+1)} - \frac{T_{n-1}'}{2(n-1)}, \quad n = 2, 3, \ldots. \tag{23}$$

Inserting this back into (20) shows the existence of a matrix $D = (d_{kl})_{0<k,l<n}$ such that

$$\sum^{n} c_k' T_k(x) = \sum^{n} \sum^{n} d_{kl} c_l T_k(x) \tag{24}$$

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$$ie. \quad \underline{c}' = D\underline{c} \tag{25}$$

where $\underline{c}'$ is the spectral representation of the derivative function u' and moreover D is a sparse upper triangular matrix, with the following properties

$$\begin{cases} d_{kl} = 0, & \text{for } k \leq l, \\ d_{kl} = 0, & \text{if } l - k \text{ is even,} \\ d_{kl} = 2k, & \text{if } l - k \text{ is odd.} \end{cases} \tag{26}$$

Applying the above result recursively, we get the spectral representation $c^{(p)}$ of the derivative with order p of u stated by

$$\underline{c}^{(p)} = D^p \underline{c}. \tag{27}$$

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For the case of integration, we recall again the relation

$$T_{n+1}(x) = nT_{n-1}(x) - 2(1-x^2)T_n(x) \quad (28)$$

$$\int T_n(x)dx = \frac{1}{2}\Big[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\Big], n = 2, 3, \quad (29)$$

$$\int T_1(x)dx = \frac{1}{4}T_2(x), \quad (30)$$

$$\int T_0(x)dx = \frac{1}{2}T_1(x). \quad (31)$$

As a linear operator, the integral of u will also be a continuous function in, which will have a unique expansion series of the form

$$\int u(x)dx = \sum_{k=0}^{n} I_k T_k(x), \qquad x \in [a, b],$$

where I_k's are coefficients of the integral of u, and similarly as with differentiation there exists a $n \times n$-matrix J such that

$$I_k = \sum_{l=0}^{n} J_{kl} c_l, \quad \text{i.e.} \quad \underline{I} = J \cdot \underline{c}, \tag{32}$$

where $\underline{I}$ is the spectral representation of the integral of u. In fact,

$$\begin{aligned} \int u(x)dx &= \int \sum_{k=0}^{N-1} c_k T_k(x) \\ i.e \qquad \sum_{k=0}^{N-1} I_k T_k(x) &= \int \sum_{k=0}^{N-1} c_k T_k(x)\, dx \\ &= \sum_{k=0}^{N-1} c_k \int T_k(x)\, dx \\ \sum_{k=0}^{N-1} \sum_{j=2}^{N-1} J_{kj} c_j T_k(x) &= \sum_{k=2}^{n} c_k \frac{1}{2} \left[\frac{T_{k+1}}{k+1} - \frac{T_{k-1}}{k-1} \right]. \end{aligned}$$

Performing a smart multiplication and rearranging terms we get the coefficients of J recursively as follows:

$$J_{kk} = 0, \quad J_{01} = \frac{1}{2}, \quad J_{k,k-1} = -J_{kk+1} = \frac{1}{k}.$$

So then, the spectral representation of the integral of u is the vector $\underline{d} = J.\underline{c}$, and for any continuous function $a(x)$,

$$\int a(x)u(x)dx \rightarrow J\,\mathbf{a}\,\underline{c}. \tag{33}$$

$$\begin{aligned} \int a_1(x)u'(x)dx &\rightarrow (\mathcal{I} - JD)\mathbf{a}_1\,\underline{c} \\ \int\int a_2(x)u''(x)dx &\rightarrow (\mathcal{I} - JD)^2\mathbf{a}_2\,\underline{c}. \end{aligned}$$

$$\vdots$$

$$\int \dots \int a_m(x) \frac{d^m u}{dx^m}(x) dx \dots dx \to (\mathcal{I} - JD)^m \mathbf{a}_m \, \underline{c} \qquad (34)$$

where $\mathcal{I}$ stands for the identity matrix. Thus, for a general linear differential operator L

$$L\, u(x) = \sum_{i=0}^{m} a_i(x) \frac{d^i u}{dx^i}(x) \qquad (35)$$

we have $$\int \dots \int L\, u(x) dx^m \to \sum_{i=0}^{m} J^{m-i} (\mathcal{I} - JD)^i \mathbf{a}_i \, \underline{c}. \qquad (36)$$

The matrix

$$A = \sum_{i=0}^{m} J^{m-i} (\mathcal{I} - JD)^i \mathbf{a}_i \qquad (37)$$

is the spectral representation of the integral operator of L.

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If we consider a general differential equation $\mathcal{A}u = f$ of order m for which the differential operator can be written as $\mathcal{A} = L + N$ where L and N are respectively the linear part and the nonlinear part, then the differential equation writes as

$$\begin{aligned} Lu(t) + Nu(t) &= f(t) && (38)\\ Lu(t) &= -Nu(t) + f(t) && (39)\\ \int \dots \int Lu(t) \to A\underline{c} &= -\mathbf{n} + J^m \underline{f} && (40)\\ A\underline{c} &= \mathbf{f} && (41)\\ \text{implying} \quad \underline{c} &= A^{-1}\mathbf{f} && (42) \end{aligned}$$

where $\mathbf{n}$ is the spectral representation of the integral of Nu at order m, and $\mathbf{f} = -\mathbf{n} + J^m \underline{f}$ is the spectral representation of $-Nu + f(t)$.

The Robust Spectral Integral Method

For this section we consider I_h to be a mesh on the interval $[0, T]$ and N be the number of subintervals and

$$I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

We denote by $\Lambda_n = [t_{n-1}, t_n],\ h_n = t_n - t_{n-1}$ and $u^n(t)$ the solution of (**??**) on the n-th element, namely

$$u^n(t) = u(t), \qquad \forall t \in \Lambda_n, \qquad 1 \leq n \leq N.$$

Let $M_n > 0$ be an integer and consider $\mathcal{P}_{M_n}$ to be the space of polynomials of order at most M_n built on Λ_n. We apply the spectral method as described previously to obtain a numerical solution $U_{M_n} \in \mathcal{P}_{M_n}$ on Λ_n. The Robust Spectral Integral Method on the interval $[0, T]$ consists of a successive application of the spectral method on each Λ_n to obtain a global numerical solution $U_M(t)$ of (**??**) defined in such way that

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For each subinterval $[t_i, t_{i+1}]$, equation (41) is applied.

$$A^{(i)}\mathbf{c}^{(i)} = \mathbf{f}^{(i)}, \qquad i = 0, ..., m-1. \tag{43}$$

The overall matrix A of the entire problem is then a diagonal of the block of matrices $A^{(i)}$.

$$\begin{pmatrix} A^{(1)} & 0 & & \\ 0 & A^{(2)} & 0 & \\ & \ddots & \ddots & \\ & & 0 & A^{(m)} \end{pmatrix} \begin{pmatrix} \mathbf{c}^{(1)} \\ \mathbf{c}^{(2)} \\ \vdots \\ \mathbf{c}^{(m)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \vdots \\ \mathbf{f}^{(m)} \end{pmatrix}. \tag{44}$$

By inversion of the matrix $A^{(i)}$ on each domain Λ_i, we obtain $c^{(i)}$ and therefore u_{M_i} which is U_M on Λ_i.

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In this case a global error can arise. However the following theorem guarantees an exponential convergence even after discretization.

Theorem

Assume that u belongs to the broken Sobolev space: $u \in H^1(0, T)$ *and* $u_{|\Lambda_n} \in H^{r_n}(\Lambda_n)$, $1 \leq n \leq N$ *with integers* $2 \leq r_n \leq M_n + 1$, *and there exists a constant* $L \geq 0$ *such that for any* z_1 *and* z_2,

$$|f(z_1, t) - f(z_2, t)| \leq L|z_1 - z_2|. \quad (45)$$

Then for

$$2\sqrt{2\pi}h_{\max}L \leq \beta < 1, \quad (46)$$

we have

$$\|u - U_M\|^2_{H^1(0,T)} \leq c_\beta T \exp(c_\beta T) \sum_{i=1}^{N} h_i^{2r_i-2} M_i^{2-2r_i} |u|^2_{H^{r_i}(\Lambda)}, \quad (47)$$

Applications and numerical results

In this section, we apply our method to different problems found in financial economics and test the convergence, and efficiency of the proposed method against the existing Chebfun method. In addition we provide an application of our method for synchronization. Since the exact solution is not available we choose the ODE15s with relative and absolute tolerance 10^{-14} to serve as the benchmark solution. The error E is the maximal error given by:

$$||E|| = ||Sol_{Benchmark} - Sol_{Numerical}||_{\infty}. \quad (48)$$

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hyperchaotic problem

It has been shown (see Zhao et al. (2011)) that four sub-blocks actually drive the dynamics of the finance model: production, money, stocks and labor force. Their interaction is reported by three nonlinear differential equations defining what is termed as the chaotic finance system. Technically and more explicitly, the finance system describes the time variation of three main state variable: the interest rate x, the investment demand y and the price index of stock z. The interest rate is an amount expressed as the percentage of the principal by lender to a borrower for an asset. The investment demand can be defined as the desired capital and inventories by firms.

hyperchotic model

The hyperchaotics finance system is expressed as follows:

$$\begin{cases} \dot{x} = z + (y - a)x + w, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \\ \dot{w} = -d\,xy - ew. \end{cases} \tag{49}$$

where the parameters a, b, c are respectively the saving, the per-investment cost and the elasticity of the demand Kocamaz et al. (2015). These parameters are all considered to be non-negative and constant.

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hyperchotic form

In order to apply our numerical methods based on spectral Chebyshev methods, we first write the system in the framework of Equation (39)). That is,

$$\begin{cases} \dot{x} + ax - z - w & = xy, \\ \dot{y} + by & = 1 - x^2, \\ \dot{z} + x + cz & = 0, \\ \dot{w} + ew & = -d\,xy, \end{cases} \tag{50}$$

hyperchaotic spectral

which can also be written as:

$$\underline{a}\, u'(t) + B\, u(t) = f(t), \quad t \in [0, T] \tag{51}$$

where $\underline{a} = (1, 1, 1, 1)$, $u(t) = [x(t), y(t), z(t), w(t)]$, $B =$
$$\begin{bmatrix} a & 0 & -1 & -1 \\ 0 & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$$
and $f(t) = [x(t)y(t), 1 - x^2(t), 0, -d\, x(t)y(t)]$.

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hyperchaotic spectral differential

The spectral representation of the Equation (51) is

$$\begin{bmatrix} D + a\mathcal{I} & 0 & -\mathcal{I} & -\mathcal{I} \\ 0 & D + b\mathcal{I} & 0 & 0 \\ Id & 0 & D + c\mathcal{I} & 0 \\ 0 & 0 & 0 & D + e\mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \\ \mathbf{f}^4 \end{bmatrix} \qquad (52)$$

where **x, y, z, w** are the spectral representation of the unknown functions $[x(t), y(t), z(t), w(t)]$ respectively, and similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3, \mathbf{f}^4]$ which represent the coefficient vectors of the nonlinear part $[xy, 1 - x^2, 0, dxy]$ respectively.

hyperchaotis spectral integral

On the other hand we can approach the hyperchaotic problem by integration first then apply Chebychev approximation to the resulting integral problem.

$$\begin{bmatrix} \mathcal{I}+aJ & 0 & -J & -J \\ 0 & \mathcal{I}+bJ & 0 & 0 \\ J & 0 & \mathcal{I}+cJ & 0 \\ 0 & 0 & 0 & \mathcal{I}+eJ \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \\ \mathbf{f}^4 \end{bmatrix} \tag{53}$$

where **x, y, z, w** are defined as above, and similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3, \mathbf{f}^4]$ which represent the coefficient vectors of the integral of the nonlinear part $[xy, 1-x^2, 0, dxy]$ respectively.

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hyperchaotic spec fix point

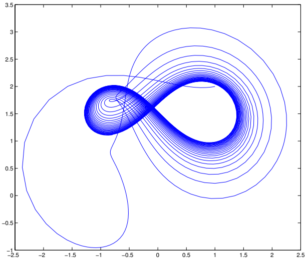
The two approaches generates nonlinear problem, we will apply an iterative method to equations (53) and (52). The aim is to get the coefficient vector $\underline{c}$ of $u(t) = [x(t), y(t), z(t)]$.
Lets then consider the fix point problem

$$A\underline{c} = \mathbf{f}. \tag{54}$$

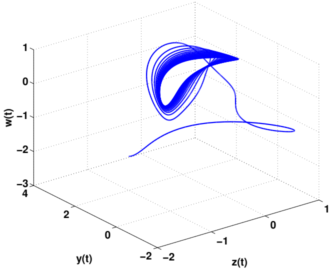
We shall start with an initial guess coming out of the initial condition then get the new $\underline{c}$ by $\mathbf{c} = A^{-1}\mathbf{f}$ where the old $\underline{c}$ is used to compute $\mathbf{f}$ in the iterations. Keeping in mind that the chaotic finance (49) is also highly nonlinear on some interval, and in order to speed up convergence we suggest the use of a splitting method on the interval $[0, T]$ into N-domains $0 = t_0 < t_1 < ... < t_N = T$ and apply the spectral methods.

phase portraits plots

The results are implemented for $a = 0.9, b = 0.2, c = 1.2, d = 0.2, e = 0.17$. Figure 1 shows the phase portraits between variables for a long time $T = 200$. They both exhibit chaos as expected.



(a) phase portrait *xy*



(b) phase portrait *yzw*

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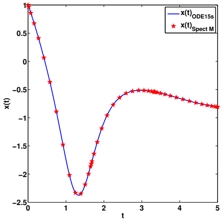
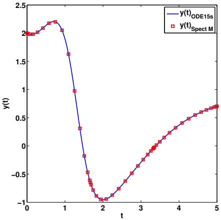
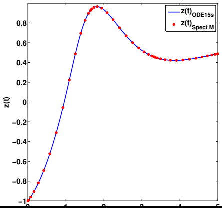
variables plots

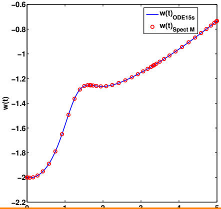


(a) $x(t)$



(b) $y(t)$

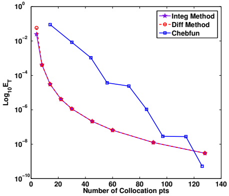




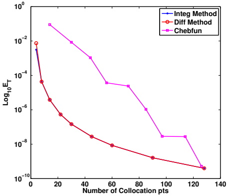
error discussion plots

We go on into investigating the effect of n and N on the decay error. Figure 3a shows that employing more collocation points on a domain enhances the precision of the numerical solution both from integration as well as differentiation method. It is also remarkable to see that while it takes close 80 points for Chebfun to reach an accuracy of 10^{-4}, the integral and differentiation spectral methods only require 20 points to achieve same accuracy. However the later methods tend to slightly lose this quality as the number of points gets larger (here $n = 120$) as compare to Chebfun. This makes us consider a domain decomposition of the interval $[0, T]$. Introducing decompositions (2 and 4 sub-intervals) the spectral decay is recovered, see Figure 3b and 3c and better accuracy is obtained.

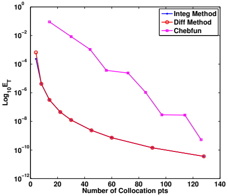
error discussion on n



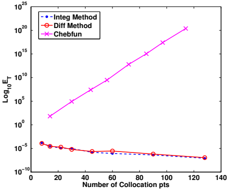
(a) Error versus n for $T = 2$ $N = 1$

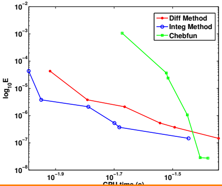


(b) Error versus n for $T = 2$ $N = 2$

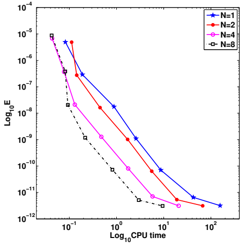


(c) Error versus n for $T = 2$ $N = 4$

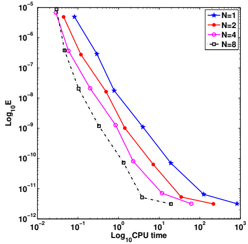




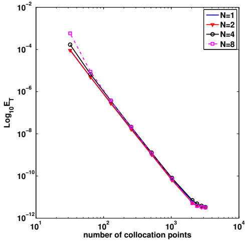
efficiency



(a) Efficiency with integration method



(b) Efficiency with differentiation method



(c) Convergence of integration/differentiation method

Figure: Efficiency and convergence of integral and differentiation method as we vary the number of domains

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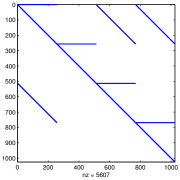
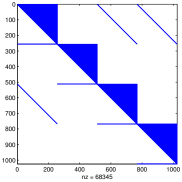
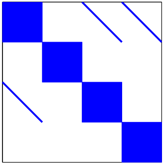
matrix plot



(a) 1-Domain Integration matrix



(b) 1-Domain Differentiation matrix



(c) Chebfun matrix

Figure: Plots of the underlying matrix A of all three methods.

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Cryptocurrency

From an asset flow perspective, Caginalp (2018) proposed a model which discribes the interaction between the market price of cryptocurrency $P(t)$, the liquidity price $L(t)$ at time t and the trend-based component of investor preference at time t denoted as $\zeta_1(t)$. This interaction is described by the following system:

$$\begin{cases} \tau_0 \frac{dP}{dt} &= (1+2\zeta_1)L - P \\ c_0 \frac{dL}{dt} &= 1 - L + q(1+2\zeta_1)L - qP \\ c_1 \frac{d\zeta_1}{dt} &= q_1(1+2\zeta_1)\frac{L}{P} - q_1 - \zeta_1 \end{cases} \tag{55}$$

The system admits only one equilibrium point obtained for $L = P$ and $\zeta_1 = 0$.

Differentiation

The differentiation approach applied in problem (55) produces the following discrete system:

$$\begin{bmatrix} \tau_0 D + \mathcal{I} & \mathcal{I} & 0 \\ q\mathcal{I} & c_0 D + (1-q)\mathcal{I} & 0 \\ 0 & 0 & c_1 D + \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \mathbf{L} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix}. \quad (56)$$

where **P, L, Z** are the coefficient vectors the variables P, L, ζ_1; similarly $[\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3]$ represent the coefficient vectors of the nonlinear part $[2\zeta_1 L, 1 + 2\zeta_1 L, q_1(1 + 2\zeta_1)\frac{L}{P}]$ respectively.

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Integration

As for the integral approach, we get the following system:

$$A\underline{c} = \mathbf{f}. \tag{57}$$

where

$$\mathbf{A} = \begin{bmatrix} \tau_0\mathcal{I} + J & -J & 0 \\ qJ & c_0\mathcal{I} + (1-q)J & 0 \\ 0 & 0 & c_1\mathcal{I} + J \end{bmatrix}$$

and **f** is the coefficient vector of the integral of the nonlinear part N,

$$N = \begin{bmatrix} 2J\zeta_1 L \\ J(1 + 2q\zeta_1 L) \\ J\left(q_1(1+2\zeta_1)\frac{L}{P} - q_1\right) \end{bmatrix}.$$

In this article, a Chebyshev spectral method has been applied on time multiple domain using differentiation matrix and also using integration approach. The methods prove to be robust with the integral approach showing to be more efficient for hyperchaotic finance problem and cryptocurrency pricing problem, than the method from differentiation approach. The results are also compared with solutions obtained from other numerical methods in the literature to confirm reliability of the solutions. The spectral methods presented here are simple, fast and accurate for handling even more complicated ODEs. For future investigation we intend to extend the spectral method designed though to the fractional case of hyperchaotic systems.

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End

Thank You!

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