$\mathbb{Z}_2^3$-graded Contractions of $\mathfrak{g}_2$

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Outline

Lie Algebras

Definition

A **Lie algebra** is a vector space L with a bilinear map

$$[-,-]\colon L \times L \to L$$

called the **Lie bracket**, satisfying:

(L1) $[x,x] = 0$ (Alternativity)

(L2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ (Jacobi Identity)

General Linear Algebra

$\mathfrak{gl}(V)$ the set of all linear maps $V \to V$ is a Lie algebra with the commutator

$$[x,y] = x \circ y - y \circ x.$$

Lie Gradings

Definition

A **grading** Γ of L

$$\Gamma\colon L = \bigoplus_{i \in I} L_i$$

is a vector space decomposition where

$$[L_i, L_j] \subseteq L_k$$

L_i is the **homogenous component of degree** i.

$\mathbb{F}[x_1, \ldots, x_n]$

Denote by $\mathbb{F}_i$ the subspace of homogenous polynomials of degree i.

$$\Gamma\colon \mathbb{F}[x_1, \ldots, x_n] = \bigoplus_{i \in \mathbb{N}} \mathbb{F}_i.$$

Group Gradings

Definition

$$\Gamma\colon L = \bigoplus_{g \in G} L_g$$

is a **group grading** if

1. Γ is indexed by a (semi-)group $(G, \cdot)$,
2. the **support** of Γ, $\{g \in G \mid L_g \neq 0\}$, generates G,
3. $[L_g, L_h] \subseteq L_{g \cdot h}$.

Group-gradings of L are **isomorphic**

$$L = \bigoplus_{g \in G} X_g = \bigoplus_{g' \in G'} Y_{g'},$$

if $\tau\colon G \cong G'$ and $f \in \mathrm{Aut}(L)$ such that $f(X_g) = Y_{\tau(g)}$.

Our Grading

Theorem

A simple $\mathbb{C}$-Lie algebra, S, of rank l admits a $\mathbb{Z}_2^{l+1}$-grading,

Applying this to $\mathfrak{g}_2 := \mathrm{Der}(\mathbb{O})$ we find a $\mathbb{Z}_2^3$-grading $\Gamma_{\mathfrak{g}_2}$ with the following properties:

1. $(\mathfrak{g}_2)_0 = 0$,
2. $[(\mathfrak{g}_2)_i, (\mathfrak{g}_2)_i] = 0$,
3. $[(\mathfrak{g}_2)_i, (\mathfrak{g}_2)_j] = (\mathfrak{g}_2)_{i+j}$

Graded Contractions

Definition

$\Gamma \colon L = \bigoplus_{g\in G} L_g$, a G-grading with G abelian. $\varepsilon \colon G \times G \to \mathbb{R}$ is a **graded contraction** if $(L, [,]^\varepsilon)$ is a Lie algebra with

$$[x, y]^\varepsilon = \varepsilon(g, h)[x, y] \ (x \in L_g, y \in L_h).$$

Examples

1. $\varepsilon(g, h) = 1, \ \forall g, h \in G : L^\varepsilon = L.$
2. $\varepsilon(g, h) = 0, \ \forall g, h \in G : L^\varepsilon$ is abelian.

Equivalences

Definition

Graded contractions ε and ε' are **Equivalent** ($\varepsilon \sim \varepsilon'$) if L^{ε} and $L^{\varepsilon'}$ are graded isomorphic.

Lemma

ε a graded contraction $\Longrightarrow$ $\sim$ **admissible** *graded contraction:*

$$\varepsilon(g,g) = \varepsilon(e,g) = 0.$$

When is $\varepsilon\colon G \times G \to \mathbb{R}$ a Graded Contraction?

$\Gamma_{\mathfrak{g}_2}$ Case

1. Anti-commutativity:

$$\varepsilon(g,h) = \varepsilon(h,g),$$

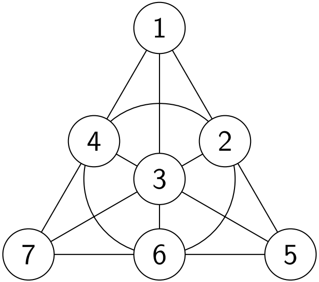
2. Jacobi identity:

$$\langle g,h,k\rangle = \mathbb{Z}_2^3 \implies \varepsilon(g,h+k)\varepsilon(h,k) = \varepsilon(k,g+h)\varepsilon(g,h).$$

$*\colon I_0 \times I_0 \to I_0$

$$g_0 := (0,0,0),\quad g_1 := (1,0,0),\quad g_2 := (0,1,0),\quad g_3 := (0,0,1),$$
$$g_4 := (1,1,1),\quad g_5 := (1,1,0),\quad g_6 := (1,0,1),\quad g_7 := (0,1,1).$$

$$I_0 := \{0,1,2,\dots,7\},\ \ X := \{\{i,j\} \mid i,j \in I,\ \ i \neq j\}$$



Admissible Graded Contraction $\leftrightarrow$ $\eta\colon X \to \mathbb{R}$

Definition

Pairwise distinct $i, j, k \in I$ are **generative** $\iff$ $i * j \neq k$ $\iff$ $\langle g_i, g_j, g_k \rangle = \mathbb{Z}_2^3$.

Set $\eta_{ijk} := n_{i,j*k}\eta_{j,k}$ for generative i, j, k.

Lemma

There is a bijection between the set of admissible graded contractions of $\Gamma_{\mathfrak{g}_2}$ and the set of maps

$$\mathcal{A} := \{\eta\colon X \to \mathbb{R} \,|\, \eta_{ijk} = \eta_{jki},\ \ i, j, k \text{ generative}\}.$$

Nice Sets

Definition

1. $i, j, k \in I$ generative, then $P_{ijk} := \{\{i,j\},\{i,k\},\{i,j*k\},\{j,k\},\{j,i*k\},\{k,i*j\}\}$.
2. $T \subseteq X$ is **nice** if $\{i,j\},\{i*j,k\} \in T \implies P_{ijk} \subseteq T$.
3. $\varepsilon \in \mathcal{A} \implies T^{\varepsilon} := \{\{i,j\} \in X \mid \varepsilon_{ij} \neq 0\}$ the **support** of ε.

Proposition

1. $\varepsilon \in \mathcal{A} \implies T^{\varepsilon}$ is nice.
2. If T is nice, then $\varepsilon^T \in \mathcal{A}$ and $T = T^{\varepsilon^T}$,

$$\varepsilon^T(t) := \begin{cases} 1, & \text{if } t \in T, \\ 0, & \text{if } t \notin T. \end{cases}$$

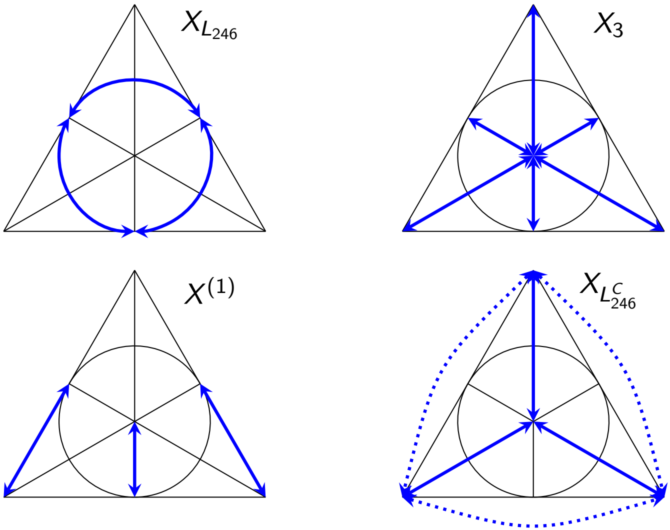
Trimming Nice Sets

Lemma

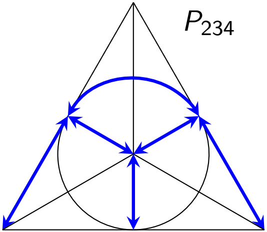
T a nice set, $\sigma \in S_*(I) := \{\sigma \in S(I) \,|\, \sigma(i * j) = \sigma(i) * \sigma(j)\}$, $\varepsilon \in \mathcal{A}$.

1. $\sigma(T)$ *is nice.*
2. $\varepsilon_\sigma \in \mathcal{A}$.
3. $\varepsilon_\sigma \sim \varepsilon$.
4. $T^{\varepsilon_\sigma} = \sigma(T^\varepsilon)$.

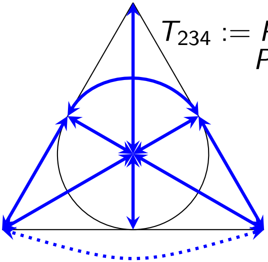
Nice Sets: Up to Collineations



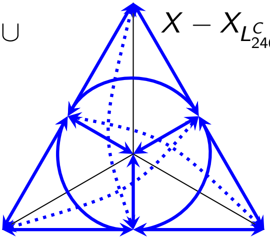
Nice Sets: Up to Collineations



P_{234}



$$T_{234} := P_{234} \cup P_{236} \cup P_{247} \cup P_{267}$$



$X - X_{L^C_{246}}$

What Next?

$$\mathfrak{b}_4, \ \mathfrak{d}_4, \ \mathfrak{f}_4, \ \mathfrak{e}_6, \ \mathfrak{e}_7, \ \mathfrak{e}_8$$