Symmetry analysis of a coupled system of differential equations

Oscar Mbusi, Ben Muatjetjeja & Abdullahi Adem

School of Mathematical Sciences
North-West University
REPUBLIC OF SOUTH AFRICA

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Preamble

(1) We perform the *Lie symmetry analysis* of the generalized Lane-Emden-Klein-Gordon-Fock system of the form

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} = 0, \; v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} = 0, \qquad (1)$$

(2) Eight cases arise for possible extension of the principal Lie algebra, which in this case is one dimensional.

(3) Moreover, symmetry reduction is carried out.

(4) Concluding remarks

Introduction

In [1], the authors, studied the coupled hyperbolic Lane-Emden system

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + v^q = 0,\; v_{tt} - v_{rr} - \frac{n}{r}v_r + u^p = 0, \tag{2}$$

with $n \neq 0$. The authors in [1] investigated both Lie and Noether point symmetries classification of (2) with the arbitrary constants $p, q \notin \{0, 1\}$ so as to bring truly nonlinearity to the system.

Motivated by the work in [1], we study the generalized Lane-Emden-Klein-Fock system of the form

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} = 0,\; v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} = 0, \tag{3}$$

where $\Phi(v)$, $\Psi(u)$ are non-zero arbitrary functions of v and u respectively.

The parameter n is assumed that it is different from 0. In fact, if we take $n = 0$, system (3) can be obtained from [2], under the complex transformation $(x, y, u, v) \longmapsto (t, ir, u, v)$ into the original variables of the aforementioned reference. Systems (2) and (3) can also be considered as natural two-component extension of the nonlinear wave equation:

$$u_{tt} - u_{rr} - \frac{n}{r} u_r - u^p = 0, \tag{4}$$

where $u = u(t, r)$ is a real-valued function, p symbolizes the interaction power and (t, r) denote time and radial coordinates respectively in $n \neq 0$ dimensions. Systems (2) and (3) are commonly encountered in many physical phenomena, see for example [1, 3, 4, 5, 7, 8] and reference therein.

In this talk we perform the Lie symmetry analysis of

the generalized Lane-Emden-Klein-Gordon-Fock system, namely

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} = 0,$$
$$v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} = 0,$$

where $\Phi(v)$ and $\Psi(u)$ are arbitrary functions of v and u respectively.

Equivalence transformations

In this section we determine the equivalence transformations [6] of system (2)-(3). The generator

$$\begin{aligned} Y &= \xi^1(t,r,u,v)\frac{\partial}{\partial t} + \xi^2(t,r,u,v)\frac{\partial}{\partial x} + \eta^1(t,r,u,v)\frac{\partial}{\partial u} \\ &\quad + \eta^2(t,r,u,v)\frac{\partial}{\partial v} + \mu^1(t,r,u,v,\Phi,\Psi)\frac{\partial}{\partial \Phi} \\ &\quad + \mu^2(t,r,u,v,\Phi,\Psi)\frac{\partial}{\partial \Psi} \end{aligned} \tag{5}$$

is said to be the generator of the equivalence group of (3) provided it is admitted by the extended system [9, 10]

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} = 0, \;\; v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} = 0, \tag{6}$$

$$\Phi_t = \Phi_r = \Phi_u = 0, \;\; \Psi_t = \Psi_r = \Psi_v = 0. \tag{7}$$

The prolongation of the generator (5) for the extended system (6)-(7) is

$$\widetilde{Y} = Y^{[2]} + \omega_t^1 \frac{\partial}{\partial \phi_t} + \omega_r^1 \frac{\partial}{\partial \phi_r} + \omega_u^1 \frac{\partial}{\partial \phi_u} + \omega_t^2 \frac{\partial}{\partial \psi_t} + \omega_r^2 \frac{\partial}{\partial \psi_r} + \omega_v^2 \frac{\partial}{\partial \psi_v}, \tag{8}$$

where $Y^{[2]}$ is the second-prolongation of (5) given by

$$\begin{aligned} Y^{[2]} &= Y + \zeta_t^1 \frac{\partial}{\partial u_t} + \zeta_r^1 \frac{\partial}{\partial u_r} + \zeta_t^2 \frac{\partial}{\partial v_t} + \zeta_r^2 \frac{\partial}{\partial v_r} + \zeta_{tt}^1 \frac{\partial}{\partial u_{tt}} \\ &\quad + \zeta_{rr}^1 \frac{\partial}{\partial u_{rr}} + \zeta_{tt}^2 \frac{\partial}{\partial v_{tt}} + \zeta_{rr}^2 \frac{\partial}{\partial v_{rr}} + \cdots . \end{aligned}$$

Here the variables ζ's and ω's are defined by the prologation formlae

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_r D_t(\xi^2), \\ \zeta_r^1 &= D_r(\eta^1) - u_t D_r(\xi^1) - u_r D_r(\xi^2), \\ \zeta_t^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_r D_t(\xi^2), \end{aligned}$$

$$
\begin{aligned}
\zeta_r^2 &= D_r(\eta^2) - v_t D_r(\xi^1) - v_r D_r(\xi^2),\\
\zeta_{tt}^1 &= D_t(\zeta_t^1) - u_{tt} D_t(\xi^1) - u_{tr} D_t(\xi^2),\\
\zeta_{rr}^1 &= D_r(\zeta_r^1) - u_{tr} D_r(\xi^1) - u_{rr} D_r(\xi^2),\\
\zeta_{tt}^2 &= D_t(\zeta_t^2) - v_{tt} D_t(\xi^1) - v_{tr} D_t(\xi^2),\\
\zeta_{rr}^2 &= D_r(\zeta_r^2) - v_{tr} D_r(\xi^1) - v_{rr} D_r(\xi^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_t^1 &= \widetilde{D}_t(\mu^1) - \phi_t \widetilde{D}_t(\xi^1) - \phi_r \widetilde{D}_t(\xi^2) - \phi_u \widetilde{D}_t(\eta^1),\\
\omega_r^1 &= \widetilde{D}_r(\mu^1) - \phi_t \widetilde{D}_r(\xi^1) - \phi_r \widetilde{D}_r(\xi^2) - \phi_u \widetilde{D}_r(\eta^1),\\
\omega_u^1 &= \widetilde{D}_u(\mu^1) - \phi_t \widetilde{D}_u(\xi^1) - \phi_r \widetilde{D}_u(\xi^2) - \phi_u \widetilde{D}_u(\eta^1),\\
\omega_t^2 &= \widetilde{D}_t(\mu^2) - \psi_t \widetilde{D}_t(\xi^1) - \psi_r \widetilde{D}_t(\xi^2) - \psi_v \widetilde{D}_t(\eta^2),\\
\omega_r^2 &= \widetilde{D}_r(\mu^2) - \psi_t \widetilde{D}_r(\xi^1) - \psi_r \widetilde{D}_r(\xi^2) - \psi_v \widetilde{D}_r(\eta^2),\\
\omega_v^2 &= \widetilde{D}_v(\mu^2) - \psi_t \widetilde{D}_v(\xi^1) - \psi_r \widetilde{D}_v(\xi^2) - \psi_v \widetilde{D}_v(\eta^2),
\end{aligned}
$$

respectively, where

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + \cdots ,$$
$$D_r = \frac{\partial}{\partial r} + u_r \frac{\partial}{\partial u} + v_r \frac{\partial}{\partial v} + \cdots ,$$

are the usual total differentiation operators and

$$\widetilde{D}_t = \frac{\partial}{\partial t} + \phi_t \frac{\partial}{\partial \phi} + \psi_t \frac{\partial}{\partial \psi} + \cdots ,$$
$$\widetilde{D}_r = \frac{\partial}{\partial r} + \phi_r \frac{\partial}{\partial \phi} + \psi_r \frac{\partial}{\partial \psi} + \cdots ,$$
$$\widetilde{D}_u = \frac{\partial}{\partial u} + \phi_u \frac{\partial}{\partial \phi} + \psi_u \frac{\partial}{\partial \psi} + \cdots ,$$
$$\widetilde{D}_v = \frac{\partial}{\partial v} + \phi_v \frac{\partial}{\partial \phi} + \psi_v \frac{\partial}{\partial \psi} + \cdots ,$$

are the new total differentiation operators for the extended system.

The invocation of the generator (8) and the invariance conditions of system (6)-(7) yields the following equivalence generators:

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \;\; X_2 = \frac{\partial}{\partial u}, \;\; X_3 = \frac{\partial}{\partial v}, \\
&X_4 = u\frac{\partial}{\partial u} + \Phi\frac{\partial}{\partial \Phi}, \;\; X_5 = v\frac{\partial}{\partial v} + \Psi\frac{\partial}{\partial \Psi}, \\
&X_6 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + (n-2)\Phi\frac{\partial}{\partial \Phi} + (n-2)\Psi\frac{\partial}{\partial \Psi}, \\
&X_7 = \frac{\partial}{\partial r} + \frac{n}{r}\Phi\frac{\partial}{\partial \Phi} + \frac{n}{r}\Psi\frac{\partial}{\partial \Psi}.
\end{aligned}
$$

Consequently, the one-parameter group of equivalence transformations corresponding to each operator is

$$
\begin{aligned}
X_1 &: \bar{t} = a_1 + t,\ \bar{r} = r, \bar{u} = u,\ \bar{v} = v,\ \bar{\Phi} = \Phi,\ \bar{\Psi} = \Psi, \\
X_2 &: \bar{t} = t,\ \bar{r} = r, \bar{u} = u + a_2,\ \bar{v} = v,\ \bar{\Phi} = \Phi,\ \bar{\Psi} = \Psi, \\
X_3 &: \bar{t} = t,\ \bar{r} = r,\ \bar{u} = u,\ \bar{v} = v + a_3,\ \bar{\Phi} = \Phi,\ \bar{\Psi} = \Psi, \\
X_4 &: \bar{t} = t,\ \bar{r} = r,\ \bar{u} = ue^{a_4},\ \bar{v} = v,\ \bar{\Phi} = \Phi e^{a_4}, \bar{\Psi} = \Psi, \\
X_5 &: \bar{t} = t,\ \bar{r} = r,\ \bar{u} = u,\ \ \bar{v} = ve^{a_5},\ \bar{\Phi} = \Phi, \bar{\Psi} = \Psi e^{a_5}, \\
X_6 &: \bar{t} = te^{a_6},\ \bar{r} = re^{a_6},\ \bar{u} = u,\ \bar{v} = v,\ \bar{\Phi} = \Phi e^{(n-2)a_6},\ \bar{\Psi} = \Psi e^{(n-2)a_6}, \\
X_7 &: \bar{t} = t,\ \bar{r} = r + a_7,\ \bar{u} = u,\ \bar{v} = v,\ \bar{\Phi} = (r + a_7)^n \frac{\Phi}{r^n}, \\
& \quad \bar{\Psi} = (r + a_7)^n \frac{\Psi}{r^n}.
\end{aligned}
$$

The composition of these transformations gives

$$
\begin{aligned}
\bar{t} &= e^{a_6}(t + a_1), \\
\bar{r} &= e^{a_6}(r + a_7), \\
\bar{u} &= e^{a_4}(u + a_2), \\
\bar{v} &= e^{a_5}(v + a_3), \\
\bar{\Phi} &= e^{a_4+(n-2)a_6}\left[(r + a_7)^n r^{-n}\Phi\right], \\
\bar{\Psi} &= e^{a_5+(n-2)a_6}\left[(r + a_7)^n r^{-n}\Psi\right].
\end{aligned}
\tag{9}
$$

Principal Lie Algebra and Group classification

Following the classical approach of group classification [10],

system (3) is invariant under the group with the generator

$$X = \xi^1(t,r,u,v)\frac{\partial}{\partial t} + \xi^2(t,r,u,v)\frac{\partial}{\partial x} + \eta^1(t,r,u,v)\frac{\partial}{\partial u} + \eta^2(t,r,u,v)\frac{\partial}{\partial v}, \quad (10)$$

if and only if

$$X^{[2]}\left(u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} = 0\right)\Big|_{(3)} = 0,$$
$$X^{[2]}\left(v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} = 0\right)\Big|_{(3)} = 0. \quad (11)$$

The expansion of (11) and separate the monomials give rise to linear overdetermined system of partial differential equations:

$$
\begin{aligned}
&\xi^1_u = 0,\ \xi^1_v = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \eta^1_{uu} = 0,\ \eta^1_{uv} = 0,\\
&\eta^1_{vv} = 0,\ \ \eta^1_{vt} = 0,\ \eta^1_{vr} = 0,\ \xi^2_r - \xi^1_t = 0,\ \xi^1_r - \xi^2_t = 0,\\
&\xi^1_{rr} - \xi^1_{tt} + \frac{n}{r}\xi^1_r + 2\eta^1_{tu} = 0,\ \ \xi^2_{rr} - \xi^2_{tt} - \frac{n}{r}\xi^2_r + \frac{n}{r^2}\xi^2 - 2\eta^1_{ru} = 0,\\
&\eta^1_{tt} - \eta^1_{rr} - \frac{n}{r}\eta^1_r - \frac{f}{r^n}\eta^1_u - \frac{g}{r^n}\eta^1_v + \frac{2f}{r^n}\xi^1_t + \frac{f'(v)}{r^n}\eta^2 - \frac{n}{r}\eta^1_r - \frac{nf}{r^{(n+1)}}\xi^1 = 0\\
&\eta^2_{uu} = 0, \eta^2_{uv} = 0,\ \eta^2_{vv} = 0,\ \eta^2_{ut} = 0,\ \eta^2_{ur} = 0,\ \xi^1_{rr} - \xi^1_{tt} + \frac{n}{r}\xi^1_r + 2\eta^2_{vt} = 0\\
&\xi^2_{rr} - \xi^2_{tt} - 2\frac{n}{r}\xi^1_t + \frac{n}{r^2}\xi^2 + \frac{n}{r}\xi^2_r - 2\eta^2_{vr} = 0,\\
&\eta^2_{tt} - \eta^2_{rr} - \frac{f}{r^n}\eta^2_u - \frac{g}{r^n}\eta^2_v + \frac{2g}{r^n}\xi^1_t - \frac{ng}{r^{(n+1)}}\xi^2 + \frac{g'(u)}{r^n}\eta^1 = 0.
\end{aligned}
$$
(12

Solving the above system of partial differential equations for arbitrary $\Phi(v)$ and $\Psi(u)$, we conclude that the system (3) admits the one-dimensional principal Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial t},$$

and the classify relations are given by

$$(\delta u + \theta)\Psi'(u) + \beta\Psi(u) + \alpha = 0, \tag{13}$$

$$(\lambda v + \gamma)\Phi'(v) + \tau\Phi(v) + \omega = 0, \tag{14}$$

where $\alpha, \beta, \gamma, \delta, \theta, \tau, \lambda$ and ω are constants. The aforementioned classifying relations are invariant under the equivalence transformations (9) if

$$\bar{\delta} = \delta, \quad \bar{\beta} = \beta, \quad \bar{\lambda} = \lambda, \quad \bar{\theta} = \delta a_2 + \theta e^{-a_4}, \quad \bar{\tau} = \tau, \quad \bar{\gamma} = \lambda a_3 + \gamma e^{-a_5},$$
$$\bar{\omega} = e^{(n-2)a_6 - a_4}\left(\frac{r^n}{(r + a_7)^n}\right), \quad \bar{\alpha} = e^{(n-2)a_6 - a_5}\left(\frac{r^n}{(r + a_7)^n}\right).$$

The invocation of the classifying relations(13) prompted the following cases for the functional forms of the arbitrary elements $\Phi(v), \Psi(u)$ and n together with the their associated extra generators. We present these results in Table 1.

Table: Classification results: $a, b, d, k, m, n, \lambda, p, q$ are constants with $p, q \neq 0$, arb=arbitrary

$\Phi(v)$ arbitrary, $\Psi(u)$ arbitrary	n arb	$X_1 = \partial_t$
$\Phi(v)$ arbitrary, $\Psi(u)$ arbitrary $X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$	$n = 2$	$X_1 = \partial_t$
$\Phi(v) = av^p$, $\Psi(u) = bu^q$, This case reduces to the system in [1].	n arb	$a, b, p, q \neq 0$

$\Phi(v) = av^{-1}, \Psi(u)$ arbitrary	n arb	$a \neq 0$
$X_1 = \partial_t,$		
$X_2 = v(n-2)\dfrac{\partial}{\partial v} - t\dfrac{\partial}{\partial t} - r\dfrac{\partial}{\partial r}$		
$\Phi(v)$ arbitrary, $\Psi(u) = bu^{-1}$	n arb	$b \neq 0$
$X_1 = \partial_t,$		
$X_2 = u(n-2)\dfrac{\partial}{\partial u} - t\dfrac{\partial}{\partial t} - r\dfrac{\partial}{\partial r}$		

$$\Phi(v) = av, \Psi(u) = bu, \quad n \text{ arb} \quad a, b \neq 0$$

$$X_1 = \partial_t,$$

$$X_2 = \frac{\partial}{\partial u},$$

$$X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

$$X_4 = av\frac{\partial}{\partial u} + bu\frac{\partial}{\partial v},$$

$$X_5 = aH\frac{\partial}{\partial u} + \left[nr^{n-1}H_r + r^n H_{rr} - r^n H_{tt}\right]\frac{\partial}{\partial v}$$

where $H(t, r)$ is any solution of partial differential equation

$$br^3(c_1 + aH) + \left[4r^{2n}n^2 - 2r^{2n}n^3 - 2r^{2n}n\right]H_r - r^{2n+3}H_{rrrr} + \left[3r^{2n+1}n - 5r^{2n+1}n^2\right]H_{rr} - 4r^{2n+2}nH_{rrr} - r^{2n+3}H_{tttt} + \left[2r^{2n+1}n^2 - r^{2n+1}n\right]H_{tt} + 4r^{2n+2}nH_{ttr} + 2r^{2n+3}H_{ttrr} = 0$$

and c_1 is an arbitrary constant.

$\Phi(v) = av, \Psi(u) = bu, \quad n = 2 \quad a, b \neq 0$

$X_1 = \partial_t,$

$X_2 = \dfrac{\partial}{\partial u},$

$X_3 = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v},$

$X_4 = av\dfrac{\partial}{\partial u} + bu\dfrac{\partial}{\partial v},$

$X_5 = t\dfrac{\partial}{\partial t} + r\dfrac{\partial}{\partial r},$

$X_6 = 2tu\dfrac{\partial}{\partial u} + 2tv\dfrac{\partial}{\partial v} - (t^2 + r^2)\dfrac{\partial}{\partial t} - 2tr\dfrac{\partial}{\partial r}$

$X_7 = aH\dfrac{\partial}{\partial u} + \left[2rH_r + r^2 H_{rr} - r^2 H_{tt}\right]\dfrac{\partial}{\partial v},$

where $H(t, r)$ satisfies the partial differential equation

$b(c_2 + aH) - 4rH_r - 14r^2 H_{rr} - 8r^3 H_{rrr} - r^4 H_{rrrr} + 6r^2 H_{tt} + 8r^3 H_{ttr}$
$-r^4 H_{tttt} + 2r^4 H_{ttrr} = 0$

and c_2 is an arbitrary constant.

$$\Phi(v) = de^{-\lambda v},\ \Psi(u) = ke^{-au} \quad n \text{ arb} \quad a, d, k, \lambda \neq 0$$
$$X_1 = \partial_t,$$
$$X_2 = \lambda(n-2)\frac{\partial}{\partial u} + a(n-2)\frac{\partial}{\partial v} - \lambda at\frac{\partial}{\partial t} - \lambda ar\frac{\partial}{\partial r}.$$

$$\Phi(v) = mv^p,\ \Psi(u) = ke^{-au} \quad n \text{ arb} \quad a, p, k, m \neq 0$$
$$X_1 = \partial_t,$$
$$X_2 = va(n-2)\frac{\partial}{\partial v} - (p+1)(n-2)\frac{\partial}{\partial u} + pat\frac{\partial}{\partial t} + par\frac{\partial}{\partial r}.$$

$$\Phi(v) = de^{-\lambda v},\ \Psi(u) = ku^q \quad n \text{ arb} \quad \lambda, d, k \neq 0$$
$$X_1 = \partial_t,$$
$$X_2 = u\lambda(n-2)\frac{\partial}{\partial u} - (q+1)(n-2)\frac{\partial}{\partial v} + \lambda qt\frac{\partial}{\partial t} + \lambda qr\frac{\partial}{\partial r}.$$

Symmetry reduction of system (3)

This section aims to transform system (3) into a system of ordinary differential equations through the invariant surface condition[11], page 169 by making use of some of the generators obtained in Table 1. First we begin with the generator $X = \partial_t$ with arbitrary $\Phi(v)$and $\Psi(u)$ and we get the two general group invariant solutions of system (3) as

$$u(t,r) = \phi(r), v(t,r) = \psi(r), \tag{15}$$

where $\phi(r)$ and $\psi(r)$ satisfy the ordinary differential system

$$\phi'' + \frac{n}{r}\phi' - \frac{\Psi}{r^n} = 0,\ \psi'' + \frac{n}{r}\psi' - \frac{\Phi}{r^n} = 0. \tag{16}$$

Other general group invariant solution of system (3) will

be obtained from the generator $X_2 = v(n-2)\frac{\partial}{\partial v} - t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r}$ with

$\Phi(v) = av^{-1}$ and $\Psi(u)$ arbitrary. Here we obtain these invariants

$$u(t,r) = \phi(z), v(t,r) = r^{-(n-2)}\psi(z), \tag{17}$$

with the similarity variable $z = \frac{t}{r}$. Substituting the values of u and v into system (3) we get

$$\begin{cases} (z^2-1)\phi'' - (n-2)z\phi' - \frac{a}{\psi} = 0, \\ (z^2-1)\psi'' + (n-2)z\psi' - (n-2)\psi + \phi = 0. \end{cases} \tag{18}$$

Thus, we conclude that

$$u(t,r) = \phi(z), v(t,r) = r^{-(n-2)}\psi(z), \tag{19}$$

is a general group invariant solution of system (3) where

ϕ and ψ are any solutions of (18).

We now work with the generator

$$X_2 = \lambda(n-2)\frac{\partial}{\partial u} + a(n-2)\frac{\partial}{\partial v} - \lambda at\frac{\partial}{\partial t} - \lambda ar\frac{\partial}{\partial r}$$

with $\Phi(v) = de^{-\lambda v}$, $\Psi(u) = ke^{-au}$ and we

obtain the three invariants

$$z = \frac{t}{r}, \;\; u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a}, \tag{20}$$

$$v(t,r) = \psi(z) + \frac{n\ln(r)}{\lambda} - \frac{2\ln(r)}{\lambda} \tag{21}$$

Invoking these invariants, system (3) transforms to

$$\begin{cases} (z^2-1)\phi'' - (n-2)z\phi' - de^{-\lambda\phi} - \frac{(n-2)(n-1)}{a} = 0, \\ (z^2-1)\psi'' - (n-2)z\psi' - ke^{-a\psi} - \frac{(n-2)(n-1)}{\lambda} = . \end{cases} \tag{22}$$

Consequently, the general group invariant solution of system (3) is

$$u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a},$$
$$v(t,r) = \psi(z) + \frac{n\ln(r)}{\lambda} - \frac{2\ln(r)}{\lambda}, \tag{23}$$

where ϕ and ψ are any solutions of the ordinary differential system (22). The other general group invariants of system (3) can also be deduced in a similar manner.

Concluding remarks

We have carried out Lie group classification of the generalized Lane-Emden-Klein-Gordon-Fock system with central symmetry (3), from the point of view of classical Lie symmetry analysis.

We found the non-equivalent forms of the functions $\Phi(v)$ and $\Psi(u)$ for which the one dimensional principal Lie algebra extends.

Some general group invariant solutions for the underlying system were constructed. It is anticipated that the results obtained in this work could be of great help in finding the solution of system (3) explicitly.

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Thank you for your attention