Congruence Lattices of Graphs

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UNIVERSITY OF JOHANNESBURG

Outline of the Talk

Lattices

Definition 1 (cf. Burris and Sankappanavar, 2012: 6)

A *lattice* is a partially ordered set (a set along with a reflexive, transitive, and antisymmetric binary relation) $\langle X, \leq \rangle$ where each pair of elements has a least upper bound, and a greatest lower bound.

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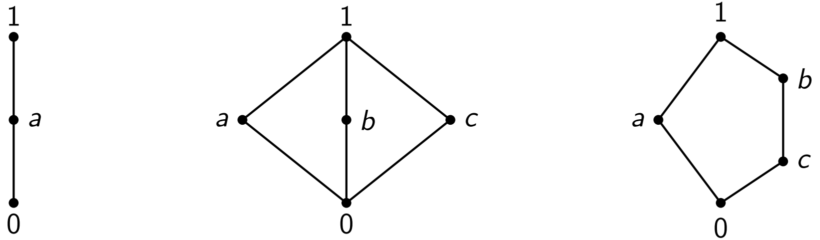
Example 2

The names of the following lattices (left to right) are: The 3-element chain, M_3, and N_5.



Distributivity and modularity

Definition 3 (cf. Burris and Sankappanavar, 2012)

A *distributive lattice* is a lattice which satisfies the following laws:

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
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A *modular lattice* is a lattice that satisfies the modular law:

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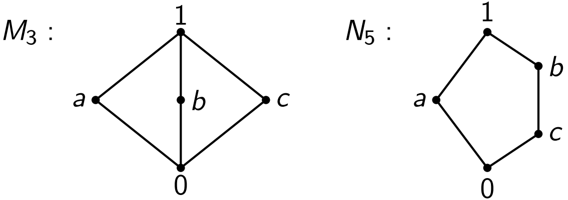
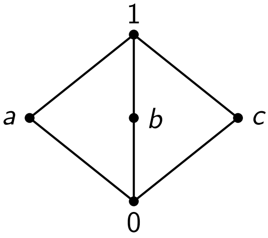


The M_3-N_5 theorem

M_3 :



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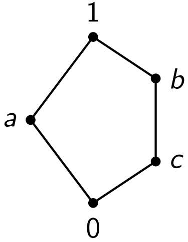

Theorem 5 (cf. Davey and Priestley, 2002)

A lattice is non-distributive if and only if it has a sublattice isomorphic to M_3 or N_5.

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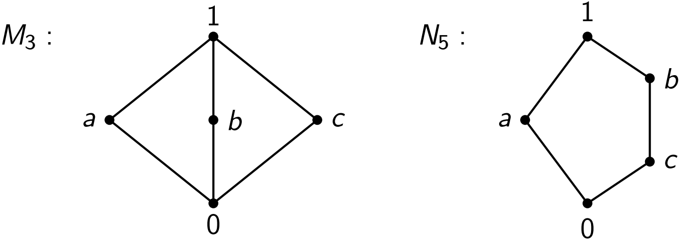

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Theorem 6 (cf. Davey and Priestley, 2002)

A lattice is non-modular if and only if it has a sublattice isomorphic to N_5.

Graph congruences

Definition 7 (Broere, Heidema and Veldsman, 2020)

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. A *congruence of* G is a pair $\theta = (\sim, \varepsilon)$ satisfying the following:
(i) $\sim$ is an equivalence relation on V_G
(ii) ε, called a congruence edge-set, is a set of unordered pairs of elements from V_G statisfying:

$$E_G \subseteq \varepsilon \subseteq \{ab | a, b \in V_G\}$$

(iii) ε statisfies the following substitution property with respect to $\sim$:
For all $x, y \in V_G$, $xy \in \varepsilon$, $x \sim a$ and $y \sim b$ implies $ab \in \varepsilon$

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Definition 8 (Broere, Heidema and Veldsman, 2020)

A congruence, $\theta = (\sim, \varepsilon)$, is *strong* if it satisfies the following:
$\varepsilon = \{xy \mid x, y \in V_G \text{ and } \exists x'y' \in E_G \text{ with } x \sim x' \text{ and } y \sim y'\}$

Graph congruence lattices

These congruences form a lattice defined as follows:

Proposition 9 (Broere, Heidema and Veldsman, 2020)

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G, and let $\mathrm{Con}(G)$ denote the set of all the congruences of G. Let, for $\theta_1 = (\sim_1, \varepsilon_1)$, $\theta_2 = (\sim_2, \varepsilon_2) \in \mathrm{Con}(G)$, the ordering is defined by:

$$\theta_1 \leq \theta_2 \text{ if and only if } \sim_1 \subseteq \sim_2 \text{ and } \varepsilon_1 \subseteq \varepsilon_2$$

Then $\langle \mathrm{Con}(G), \leq \rangle$ forms a lattice.

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$$\sim_1 = \{\langle a, a\rangle, \langle b, b\rangle\} \quad \text{and} \quad \sim_2 = \{\langle a, a\rangle, \langle b, b\rangle, \langle a, b\rangle, \langle b, a\rangle\}$$

Congruence lattice example 1

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$\sim_1 = \{\langle a, a\rangle, \langle b, b\rangle\}$ and $\sim_2 = \{\langle a, a\rangle, \langle b, b\rangle, \langle a, b\rangle, \langle b, a\rangle\}$

Substitution property: If $xy \in \varepsilon$, $x \sim a$ and $y \sim b$, then $ab \in \varepsilon$

The congruence edge sets corresponding to $\sim_1$ are:

$\varepsilon_{11} = \{aa\}$, $\varepsilon_{12} = \{aa, bb\}$, $\varepsilon_{13} = \{aa, ab\}$, and $\varepsilon_{14} = \{aa, bb, ab\}$

The congruence edge set corresponding to $\sim_2$ is $\varepsilon_{21} = \{aa, ab, bb\}$

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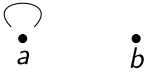

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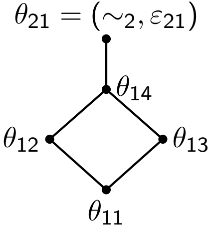

Congruence lattice example 2

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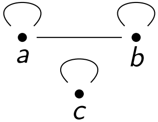

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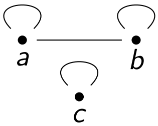

$\sim_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$, $\sim_2 = \sim_1 \cup \{\langle a, b\rangle, \langle b, a\rangle\}$, $\sim_3 = \sim_1 \cup \{\langle a, c\rangle, \langle c, a\rangle\}$, $\sim_4 = \sim_1 \cup \{\langle b, c\rangle, \langle c, b\rangle\}$, and $\sim_5 = \sim_1 \cup \{\langle a, b\rangle, \langle b, a\rangle, \langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$

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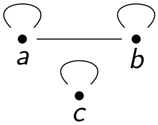

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$$\sim_3 = \sim_1 \cup \{\langle a,c\rangle,\langle c,a\rangle\},\ \sim_4 = \sim_1 \cup \{\langle b,c\rangle,\langle c,b\rangle\}, \text{ and}$$
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The congruence edge-sets corresponding to $\sim_1$ are:

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The congruence edge-sets corresponding to $\sim_2$ are:

$$\varepsilon_{21} = \{aa, bb, cc, ab\}, \text{ and } \varepsilon_{22} = \{aa, bb, cc, ab, ac, bc\}$$

Graph congruence example 2 continued

The congruence edge-set corresponding to $\sim_3$ is:

$$\varepsilon_{31} = \{aa, bb, cc, ab, ac, cb\}$$

The congruence edge-sets corresponding to $\sim_4$ and $\sim_5$, ε_{41} and ε_{51}, respectively, are the same as ε_{31}

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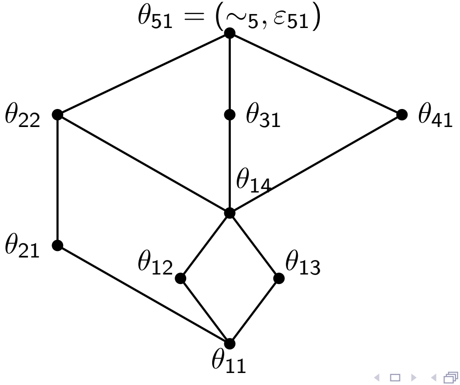

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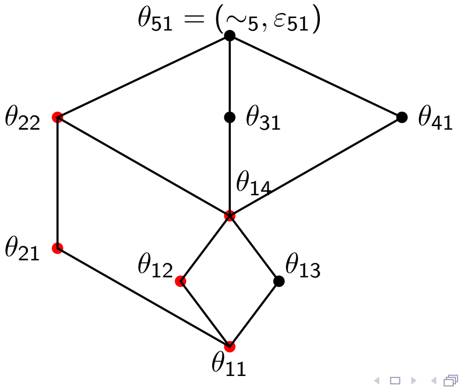

Partition lattices

Definition 10 (cf. Grätzer, 2011: Section 4.1)

If A is a non-empty set, then the *partition lattice* (or the *equivalence lattice*) of A is the lattice whose elements are the equivalence relations of A, ordered by set inclusion.

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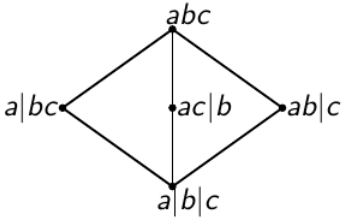

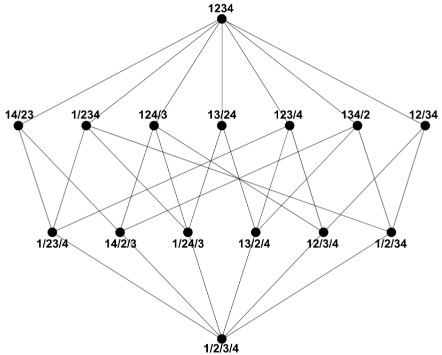

Partition lattices and graph congruence lattices

Lemma 11

Let P be a sublattice of P_n, the partition lattice for n elements, and let G be a graph with n vertices. Then there is an isomorphism from P to a sublattice of $\mathrm{Con}(G)$.

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Proof outline:

- Let $S \subseteq \text{Con}(G)$ contain all equivalence relations in P, paired with the congruence-edge set containing all possible edges, ε_1.
- Show S is a sublattice of $\text{Con}(G)$.

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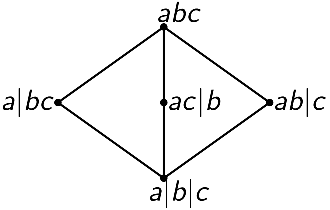
- The map $\sigma : P \to S$, as $\sim_a \mapsto (\sim_a, \varepsilon_1)$ is a lattice isomorphism.

When are graph congruence lattices distributive?

The partition lattice for a three element set $\{a, b, c\}$ is:



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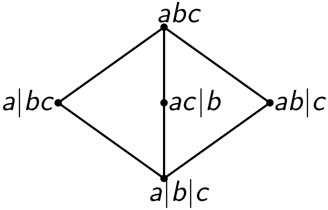

Proposition 12

For any natural number n, the partition lattice of n elements, P_n, is isomorphic to a sublattice of the partition lattice of $n+1$ elements, P_{n+1}.

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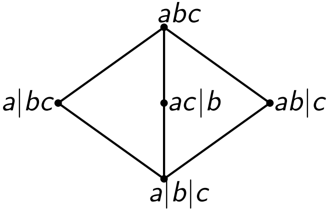

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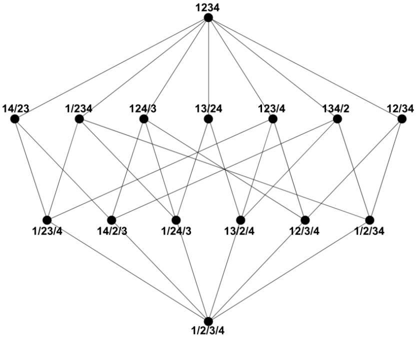
The congruence lattice of any graph with three or more vertices is non-distributive.

When are graph congruence lattices modular?

The partition lattice for a 4 element set $\{1, 2, 3, 4\}$ is:



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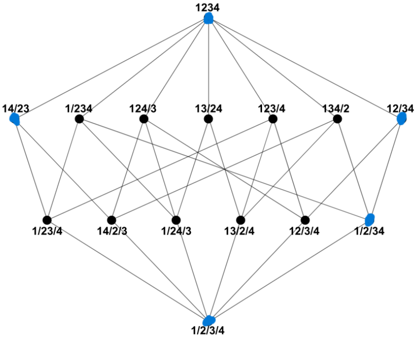

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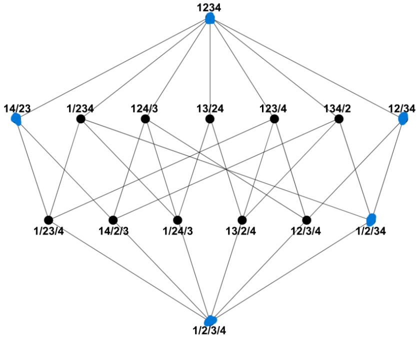

Theorem 14

The congruence lattice of any graph with four or more vertices is non-modular.

Identites of congruence lattices

Theorem 15 (Pudlák and Tůma, 1980)

Every finite lattice, L, can be embedded in a finite partition lattice, P_n.

Lemma 16 (cf. Grätzer, 2011: Lemma 59)

Identities are preserved under the formation of sublattices.

Theorem 17

There is no non-trivial lattice identity that all graph congruence lattices satisfy.

Structure of graph congruence lattices

Theorem 18

Let G be a graph with n vertices, and let $\sim_m$ be an equivalence relation on V_G. The then set of congruences on G with equivalence relation $\sim_m$ forms a boolean sublattice of $\mathrm{Con}(G)$.

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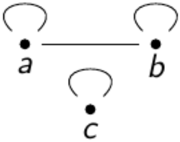
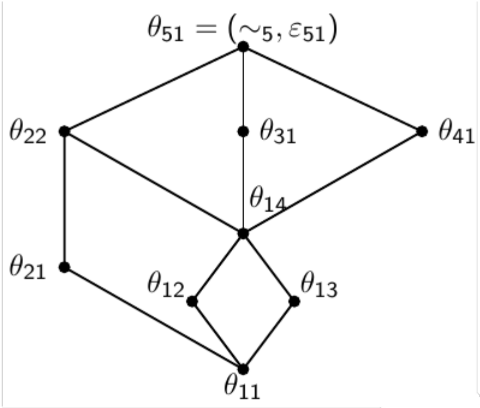
- Show that $\{\varepsilon_{m1}, \varepsilon_{m2}, ..., \varepsilon_{mk}\} = \{\varepsilon_{ms} \cup (\cup_{i \in S} C_i) | S \in \mathcal{P}(\{1, ..., p\})\}$.
- The map $\varepsilon_{mx} \mapsto (\sim_m, \varepsilon_{mx})$ is then an isomorphism from a boolean lattice to the congruences with equivalence relation $\sim_m$.

Structure of graph congruence lattices





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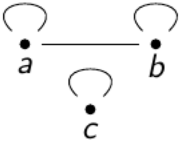

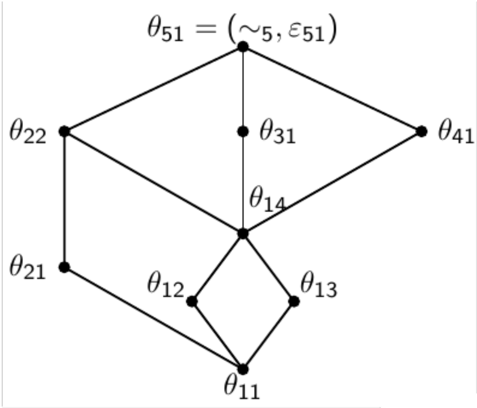

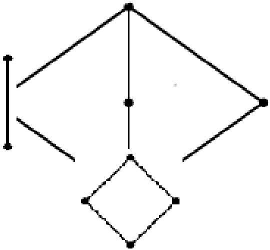

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What we have achieved:

- Provided necessary conditions for modularity and distributivity for congruence lattices
- Shown that there are no non-trivial lattice identities statisfied by all congruence lattices
- Explained the basic structure of congruence lattices

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- Provided necessary conditions for modularity and distributivity for congruence lattices
- Shown that there are no non-trivial lattice identities statisfied by all congruence lattices
- Explained the basic structure of congruence lattices

What can be done from here:

- Investigate the addition of edges
- Investigate what characteristics lead to non-modularity for 3 element graphs
- Invesigate non-identities (such as semi-modularity and semi-distributivity)
- Investigate the properties of the congruence algebras of graphs without loops

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