Generalized Iterated Function System for Common Attractors in Partial Metric Spaces

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Outline

Introduction

The mathematical foundations of iterated function system (IFS), were laid down in 1981 by Hutchinson [1]. He proved that the Hutchinson operator defined on $\mathbb{R}^k$ has as a fixed point, a set in $\mathbb{R}^k$ which is closed and bounded, known as an attractor of IFS [2].

[1] J. Hutchinson, Fractals and self-similarity. Indiana Univ. J. Math, 30(5) (1981), 713-747

[2] T. Nazir, S. Silverstrov, M. Abbas, Fractals of generalized F-Hutchinson operator. Waves Wavelets Fractals Adv. Anal, 2 (2016), 29-40.

In this talk, we construct some new common attractors with the assistance of generalized Hutchinson contractive operator, defined on a complete partial metric space. We observe that this (Hutchinson) operator is itself a generalized contractive mapping on a finite family of compact subsets of a space say Y. By successive application of a generalized Hutchinson operator, a final fractal [1] is obtained. This shall be followed by a presentation of a nontrivial example in support of the proven results.

[1] M.F. Barnsley, Fractals Everywhere, 2nd ed. Academic Press, San Diego, CA (1993).

Preliminaries

Definition 1.1. [1]. A partial metric on a non-empty set Y is a mapping $p : Y \times Y \to \mathbb{R}^+$ with the following properties:

(p_1) $p(z_1, z_1) = p(z_1, z_2) = p(z_2, z_2)$, if and only if $z_1 = z_2$,

(p_2) $p(z_1, z_1) \leq p(z_1, z_2)$,

(p_3) $p(z_1, z_2) = p(z_2, z_1)$,

(p_4) $p(z_1, z_2) \leq p(z_1, z_3) + p(z_3, z_2) - p(z_3, z_3)$,

for all $z_1, z_2, z_3 \in Y$. The pair (Y, p) consisting of the set Y and the partial metric p, is called a **partial metric space**.

[1] S. G. Matthews, Partial metric topology, In: Proceedings ofthe 8th Summer Conference on General Topology and Applications. Annals of New York Academy of Sciences, 728 (1994), 183-197

Example 1.2.[1]
A pair $(\mathbb{R}^+, p)$, with $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $p(z_1, z_2) = \max\{z_1, z_2\}$ for all $z_1, z_2 \in \mathbb{R}^+$ is an example of a partial metric space.

Definition 1.3.[2 3] Consider a partial metric space (Y, p). Then

(i) $\{z_k\}$ is called a Cauchy sequence in Y if $\lim_{k,\eta \to +\infty} p(z_k, z_\eta)$ exists and is finite.

(ii) (Y, p) is said to be complete if every Cauchy sequence $\{z_k\}$ in Y converges to a point $z \in Y$ such that $p(z, z) = \lim_{k \to +\infty} p(z_k, z)$.

[1] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, Topology Appl., 159 (2012), 3234-3242.

[2] S. G. Matthews, Partial metric topology, In: Proceedings of the 8th Summer Conference on General Topology and Applications, Annals of New

Definition 1.4. [1] Let (Y, p) be a partial metric space and $\mathcal{C}^p \subseteq Y$. Then $\mathcal{C}^p$ is said to be compact if every sequence $\{v_n\}$ in $\mathcal{C}^p$ contains a subsequence $\{v_{n_i}\}$ which converges to a point in $\mathcal{C}^p$.

[1] In a partial metric space (Y, p), let $\mathcal{C}^p(Y)$ denote the set of all non-empty compact subsets of Y. For $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$, let

$$H_p(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\eta \in \mathcal{N}} p(\eta, \mathcal{M}), \sup_{\mu \in \mathcal{M}} p(\mu, \mathcal{N})\},$$

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where $p(z, \mathcal{M}) = \inf\{p(z, \mu) : \mu \in \mathcal{M}\}$ is a measure of how far a point z is from the set $\mathcal{M}$. Such a mapping H_p is referred to as the Pompeiu-Hausdorff metric induced by the partial metric p. $(\mathcal{C}^p(Y), H_p)$ is a complete partial metric space, provided (Y, p) is a complete partial metric space.

Proposition 1.5. Let (Y, p) be a partial metric space. Then for all $L, \mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$,

(a) $H_p(\mathcal{L}, \mathcal{L}) \leq H_p(\mathcal{L}, \mathcal{M})$,

Proposition 1.5. Let (Y, p) be a partial metric space. Then for all $L, \mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$,

(a) $H_p(\mathcal{L}, \mathcal{L}) \leq H_p(\mathcal{L}, \mathcal{M})$,

(b) $H_p(\mathcal{L}, \mathcal{M}) = H_p(\mathcal{M}, \mathcal{L})$,

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(b) $H_p(\mathcal{L}, \mathcal{M}) = H_p(\mathcal{M}, \mathcal{L})$,

(c) $H_p(\mathcal{L}, \mathcal{M}) \leq H_p(\mathcal{L}, \mathcal{N}) + H_p(\mathcal{N}, \mathcal{M}) - \inf_{\eta \in \mathcal{N}} p(\eta, \eta)$.

Proposition 1.5. Let (Y, p) be a partial metric space. Then for all $L, \mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$,

(a) $H_p(\mathcal{L}, \mathcal{L}) \leq H_p(\mathcal{L}, \mathcal{M})$,

(b) $H_p(\mathcal{L}, \mathcal{M}) = H_p(\mathcal{M}, \mathcal{L})$,

(c) $H_p(\mathcal{L}, \mathcal{M}) \leq H_p(\mathcal{L}, \mathcal{N}) + H_p(\mathcal{N}, \mathcal{M}) - \inf_{\eta \in \mathcal{N}} p(\eta, \eta)$.

Theorem 1.6. [1] In a complete partial metric space (Y, p), let $h : Y \to Y$ be a contraction mapping such that, for any $\lambda \in [0, 1)$,

$$p\left(hz_1, hz_2\right) \leq \lambda p(z_1, z_2),$$

is true for all $z_1, z_2 \in Y$. Then there exists a unique fixed point u of h in Y and for every v_0 in Y, the sequence $\{v_0, hv_0, h^2 v_0, ...\}$ converges to the fixed point u of h.

[1] S. G. Matthews, Partial metric topology, In: Proceedings of the 8th Summer Conference on General Topology and Applications. Annals of New York Academy of Sciences, 728 (1994), 183-197.

Theorem 1.7. Consider a partial metric space (Y, p). Let $\{h_k : k = 1, 2, ..., r\}$ be a finite collection of contraction mappings on Y with contraction constants $\lambda_1, \lambda_2, ..., \lambda_r$, respectively. Define $\Psi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ by

$$\begin{aligned}\Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_r(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}),\end{aligned}$$

for each $\mathcal{M} \in \mathcal{C}^p(Y)$. Then Ψ is a contraction mapping on $\mathcal{C}^p(Y)$ with contraction constant $\lambda = \max\{\lambda_1, \lambda_2, ..., \lambda_r\}$

Definition 1.8.[1] Let (Y, p) be a complete partial metric space. If $h_k : Y \to Y$, for each $k = 1, 2, ..., r$ are contraction mappings, then $\{Y; h_k, k = 1, 2, \cdots, r\}$ is called an **iterated function system** (IFS).

[1]T. Nazir, S. Silverstrov and M. Abbas, Fractals of generalized F-Hutchinson operator, Waves Wavelets Fractals Adv. Anal., 2 (2016), 29-40.

Definition 1.9. [1] Let $\mathcal{M} \subseteq Y$ be a non-empty compact set, then $\mathcal{M}$ is an **attractor** of the IFS if

(i) $\Psi(\mathcal{M}) = \mathcal{M}$ and

(ii) there exist an open set $V_1 \subseteq Y$ such that $\mathcal{M} \subseteq V_1$ and $\lim_{k \to +\infty} \Psi^k(\mathcal{N}) = \mathcal{M}$ for any compact set $\mathcal{N} \subseteq V_1$, where the limit is taken with respect to the partial Hausdorff metric.

The maximal open set V_1 such that (ii) is satisfied is called a basin of attraction.

[1] T. Nazir, S. Silverstrov and M. Abbas, Fractals of generalized F-Hutchinson operator, Waves Wavelets Fractals Adv. Anal., 2 (2016), 29-40.

Generalized iterated function system

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for all $z_1, z_2 \in Y$, where $\lambda \in [0, 1)$.

Theorem 2.2. Let (Y, p) be a partial metric space and $h, g : Y \to Y$ be two continuous mappings. If the pair (h, g) is a generalized contraction with $\lambda \in [0, 1)$, then

(1) the elements in $\mathcal{C}^p(Y)$ are mapped to elements in $\mathcal{C}^p(Y)$ under h and g;

(2) if for any $U \in \mathcal{C}^p(Y)$, the mappings $h, g : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ defined as

$$\begin{aligned} h(U) &= \{h(z_1) : z_1 \in U\} \text{ and} \\ g(U) &= \{g(z_2) : z_2 \in U\}, \end{aligned}$$

then the pair (h, g) is a generalized contraction on $(\mathcal{C}^p(Y), H_p)$.

Proposition 2.3. In a partial metric space (Y, p), suppose the mappings $h_k, g_k : Y \to Y$ for $k = 1, 2, \cdots, r$ are continuous and satisfy

$$p(h_k z_1, g_k z_2) \leq \lambda_k p(z_1, z_2) \text{ for all } z_1, z_2 \in Y,$$

where $\lambda_k \in [0, 1)$ for each $k \in \{1, 2, \cdots, r\}$. Then the mappings $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ defined as

$$\begin{aligned} \Psi(U) &= h_1(U) \cup h_2(U) \cup \cdots \cup h_r(U) \\ &= \cup_{k=1}^r h_k(U) \text{ for each } U \in \mathcal{C}^p(Y) \end{aligned}$$

and

$$\begin{aligned}\Phi(V) &= g_1(V) \cup g_2(V) \cup \cdots \cup g_r(V) \\ &= \cup_{k=1}^{r} g_k(V) \text{ for each } V \in \mathcal{C}^p(Y)\end{aligned}$$

also satisfy

$$H_p(\Psi U, \Phi V) \leq \widetilde{\lambda} H_p(U, V) \text{ for all } U, V \in \mathcal{C}^p(Y),$$

where $\widetilde{\lambda} = \max\{\lambda_k : k \in \{1, 2, ..., r\}\}$. The pair (Ψ, Φ) is a generalized contraction on $\mathcal{C}^p(Y)$.

Definition 2.4. In a partial metric space (Y, p) with the mappings $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$, a pair of mappings (Ψ, Φ) is called

(1) a **generalized Hutchinson contractive operator** if a constant $\lambda \in [0, 1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$, the following holds:

$$H_p\left(\Psi\left(\mathcal{M}\right), \Phi\left(\mathcal{N}\right)\right) \leq \lambda Z_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

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$$H_p\left(\Psi\left(\mathcal{M}\right), \Phi\left(\mathcal{N}\right)\right) \leq \lambda Z_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}) = {} & \max\{H_p(\mathcal{M}, \mathcal{N}), H_p(\mathcal{M}, \Psi\left(\mathcal{M}\right)), \\ & H_p(\mathcal{N}, \Phi\left(\mathcal{N}\right)), \\ & \frac{H_p(\mathcal{M}, \Phi\left(\mathcal{N}\right)) + H_p(\mathcal{N}, \Psi\left(\mathcal{M}\right))}{2}\}. \end{aligned}$$

(2) a **generalized rational Hutchinson contractive operator** if a constant $\lambda_* \in [0,1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$, the following holds:

$$H_p\left(\Psi\left(\mathcal{M}\right), \Phi\left(\mathcal{N}\right)\right) \leq \lambda_* R_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

where

(2) a **generalized rational Hutchinson contractive operator** if a constant $\lambda_* \in [0,1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$, the following holds:

$$H_p\left(\Psi\left(\mathcal{M}\right), \Phi\left(\mathcal{N}\right)\right) \leq \lambda_* R_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

where

$$\begin{aligned} R_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}) \quad = \quad & \max\left\{ \frac{H_p(\mathcal{M}, \Phi\left(\mathcal{N}\right))[1 + H_p(\mathcal{M}, \Psi(\mathcal{M}))]}{2(1 + H_p\left(\mathcal{M}, \mathcal{N}\right))}, \right. \\ & \frac{H_p(\mathcal{N}, \Phi\left(\mathcal{N}\right))[1 + H_p(\mathcal{M}, \Psi(\mathcal{M}))]}{1 + H_p\left(\mathcal{M}, \mathcal{N}\right)}, \\ & \left. \frac{H_p(\mathcal{M}, \mathcal{N})[1 + H_p(\mathcal{M}, \Psi\left(\mathcal{M}\right))]}{1 + H_p(\mathcal{M}, \mathcal{N})} \right\}. \end{aligned}$$

Main results

Theorem 3.1. Let (Y, p) be a complete partial metric space and $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$ be a generalized iterated function system. Let $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ be defined by

$$\Psi(\mathcal{M}) = \cup_{k=1}^{r} h_k(\mathcal{M}),$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{r} g_k(\mathcal{N})$$

for each $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$. If the pair (Ψ, Φ) is a generalized Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $U_1 \in \mathcal{C}^p(Y)$, that is,

$$U_1 = \Psi(U_1) = \Phi(U_1).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{M}_0 \in \mathcal{C}^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \ldots\}$$

of compact sets converges to the common attractor U_1 of Ψ and Φ.

Proof. See [1]

[1] M. Khumalo, T. Nazir, V. Makhoshi, Generalized iterated function system for common attractors in partial metric spaces, AIMS Mathematics, 7(7) (2022) 13074-13103.

Theorem 3.2. (**Generalized Collage**) Let (Y, p) be a complete partial metric space. For a given generalized iterated function system $\{Y; h_1, h_2..., h_r; g_1, g_2, ..., g_r\}$ which have contractive constant $\lambda \in [0, 1)$ and for a given $\varepsilon \geq 0$, if for any $\mathcal{M} \in \mathcal{C}^p(Y)$, we have either

$$H_p(\mathcal{M}, \Psi(\mathcal{M})) \leq \varepsilon,$$

or

$$H_p(\mathcal{M}, \Phi(\mathcal{M})) \leq \varepsilon,$$

where $\Psi(\mathcal{M}) = \cup_{k=1}^{r} h_k(\mathcal{M})$ and $\Phi(\mathcal{M}) = \cup_{k=1}^{r} g_k(\mathcal{M})$. Then,

$$H_p(\mathcal{M}, U_1) \leq \frac{\varepsilon}{1-\lambda},$$

where $U_1 \in \mathcal{C}^p(Y)$ is a common attractor of Ψ and Φ.
Proof. See [1]

[1] M. Khumalo, T. Nazir, V. Makhoshi, Generalized iterated function system for common attractors in partial metric spaces, AIMS Mathematics, 7(7) (2022) 13074-13103.

Remark 3.3. If we take in Theorem 3.1, $\mathcal{S}^p(Y)$ the collection of all singleton subsets of the given space Y, then $\mathcal{S}^p(Y) \subseteq \mathcal{C}^p(Y)$. Furthermore, if we take a pair of mappings $(h_k, g_k) = (h, g)$ for each k, where $h = h_1$ and $g = g_1$ then the pair of operators (Ψ, Φ) becomes

$$(\Psi(z_1), \Phi(z_2)) = (h(z_1), g(z_2)).$$

Consequently, the following common fixed point result is obtained.

Corollary 3.4. Suppose $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$ is a generalized iterated function system defined in a complete partial metric space (Y, p) and define a pair of mappings $h, g : Y \to Y$ as in Remark 3.3. If some $\lambda \in [0, 1)$ exists such that for any $z_1, z_2 \in Y$, the following condition holds:

$$p(hz_1, gz_2) \leq \lambda Z_{h,g}(z_1, z_2),$$

where

$$\begin{aligned} Z_{h,g}(z_1, z_2) &= \max\{p(z_1, z_2), p(z_1, hz_1), p(z_2, gz_2), \\ &\quad \frac{p(z_1, gz_2) + p(z_2, hz_1)}{2}\}. \end{aligned}$$

Then h and g have a unique common fixed point $u \in Y$. Furthermore, for any $u_0 \in Y$, the sequence $\{u_0, hu_0, ghu_0, hghu_0, \cdots\}$ converges to the common fixed point of h and g.

Corollary 3.5. Let $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$ be a generalized iterated function system defined in a complete partial metric space (Y, p) and (h_k, g_k) for $k = 1, 2, ..., r$ be a pair of generalized contractive self-mappings on Y. Then the pair $(\Psi, \Phi) : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ defined in Theorem 3.1 has a unique common attractor in $\mathcal{C}^p(Y)$. Furthermore, for any initial set $\mathcal{M}_0 \in \mathcal{C}^p(Y)$, the sequence $\{\mathcal{M}_0, \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \cdots\}$ of compact sets converges to the common attractor of both Ψ and Φ.

Example 3.6. Let $Y = [0, 10]$ be endowed with the partial metric $p : Y \times Y \to \mathbb{R}^+$ defined by

$$p(z_1, z_2) = \frac{1}{2}\max\{z_1, z_2\} + \frac{1}{4}|z_1 - z_2| \text{ for all } z_1, z_2 \in Y.$$

Define $h_1, h_2 : Y \to Y$ as

$$\begin{aligned} h_1(z) &= \frac{10 - z}{3} \text{ for all } z \in Y, \\ h_2(z) &= \frac{16 - z}{4} \text{ for all } z \in Y \end{aligned}$$

and $g_1, g_2 : Y \to Y$ as

$$\begin{aligned} g_1(z) &= \frac{15 - z}{3} \text{ for all } z \in Y, \\ g_2(z) &= \frac{z + 4}{4} \text{ for all } z \in Y. \end{aligned}$$

Now, for $z_1, z_2 \in Y$, we have

$$\begin{aligned} p\left(h_1\left(z_1\right), g_1\left(z_2\right)\right) &= \frac{1}{2}\max\left\{\frac{10-z_1}{3}, \frac{15-z_2}{3}\right\} \\ &+ \frac{1}{4}\left|\frac{10-z_1}{3} - \frac{15-z_2}{3}\right| \\ &= \frac{1}{3}\left[\frac{1}{2}\max\{10-z_1, 15-z_2\}\right. \\ &+ \left.\frac{1}{4}\left|(10-z_1)-(15+z_2)\right|\right] \\ &\leq \lambda_1 p\left(z_1, z_2\right), \end{aligned}$$

where $\lambda_1 = \dfrac{1}{3}$.

Also, for $z_1, z_2 \in Y$, we have

$$\begin{aligned}
p\left(h_2\left(z_1\right), g_2\left(z_2\right)\right) &= \frac{1}{2}\max\left\{\frac{16-z_1}{4}, \frac{z_2+4}{4}\right\} \\
&+ \frac{1}{4}\left|\frac{16-z_1}{4} - \frac{z_2+4}{4}\right| \\
&= \frac{1}{4}\left[\frac{1}{2}\max\{16-z_1, z_2+4\}\right. \\
&+ \left.\frac{1}{4}\left|(16-z_1)-(z_2+4)\right|\right] \\
&\leq \lambda_2 p\left(z_1, z_2\right),
\end{aligned}$$

where $\lambda_2 = \dfrac{1}{4}$.

Consider the generalized iterated function system $\{Y; (h_1, g_1), (h_2, g_2)\}$ with the mappings $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ given as

$$(\Psi, \Phi)(U) = (h_1, g_1)(U) \cup (h_2, g_2)(U) \text{ for all } U \in \mathcal{C}^p(Y).$$

By Proposition 2.3, for $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$, we have

$$H_p(\Psi(\mathcal{M}), \Phi(\mathcal{N})) \leq \lambda^* H_p(\mathcal{M}, \mathcal{N}),$$

where $\lambda^* = \max\left\{\frac{1}{3}, \frac{1}{4}\right\} = \frac{1}{3}$. Thus, all conditions of Corollary 3.5 are satisfied. Moreover, for any initial set $\mathcal{M}_0 \in \mathcal{C}^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \cdots\}$$

of compact sets is convergent and has a limit point which is the common attractor of Ψ and Φ. □

Theorem 3.7. Consider a complete partial metric space (Y, p) and the generalized iterated function system given as $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$. Let $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ be defined by

$$\Psi(\mathcal{M}) = \cup_{k=1}^{r} h_k(\mathcal{M})$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{r} g_k(\mathcal{N}),$$

for each $\mathcal{M}, \mathcal{N} \in \mathcal{C}^p(Y)$. If the pair (Ψ, Φ) is generalized rational Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $U_1 \in \mathcal{C}^p(Y)$, that is,

$$U_1 = \Psi(U_1) = \Phi(U_1).$$

Furthermore, for arbitrarily chosen initial set $\mathcal{M}_0 \in \mathcal{C}^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \cdots\}$$

of compact sets converges to a common attractor U_1.

Corollary 3.8. Consider a generalized iterated function system $\{Y; h_k, g_k, k = 1, 2, \cdots, r\}$ on a complete partial metric space (Y, p) and the mappings $h, g : Y \to Y$ as given in Remark 3.3. If there exists $\lambda_* \in [0, 1)$ such that for any $z_1, z_2 \in Y$, the following condition holds:

$$p\left(hz_1, gz_2\right) \leq \lambda_* R_{h,g}(z_1, z_2),$$

where

$$
\begin{aligned}
R_{h,g}(z_1, z_2) \quad = \quad & \max\left\{ \frac{p(z_1, gz_2)[1 + p(z_1, hz_1)]}{2(1 + p(z_1, zy_2))}, \right. \\
& \frac{p(z_2, gz_2)[1 + p(z_1, hz_1)]}{1 + p(z_1, z_2)}, \\
& \left. \frac{p(z_1, z_2)[1 + p(z_1, hz_1)]}{1 + p(z_1, z_2)} \right\}.
\end{aligned}
$$

Then a unique common fixed point for h and g exists. Furthermore, for any initial choice of $v_0 \in Y$, the sequence $\{v_0, hv_0, ghv_0, hghv_0, ...\}$ converges to the common fixed point of h and g.

Well-posedness of Common attractor based problem

Definition 4.1. [1] A common attractor-based problem of a pair of mappings $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ is said to be well-posed if the pair (Ψ, Φ) has a unique common attractor $\Lambda_* \in \mathcal{C}^p(Y)$ and for any sequence $\{\Lambda_k\}$ in $\mathcal{C}^p(Y)$ such that $\lim_{k\to+\infty} H_p(\Psi(\Lambda_k), \Lambda_k) = 0$ and $\lim_{k\to+\infty} H_p(\Phi(\Lambda_k), \Lambda_k) = 0$, then $\lim_{k\to+\infty} H_p(\Lambda_k, \Lambda_*) = H_p(\Lambda_*, \Lambda_*)$, that is, $\lim_{k\to+\infty} \Lambda_k = \Lambda_*$.

[1] M. A. Kutbi, A. Latif and T. Nazir, Generalized rational contractions in semi metric spaces via iterated function system, RACSAM, 114:187 (2020), 1-16.

Theorem 4.2. Let (Y, p) be a complete partial metric space and $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ be defined as in Theorem 3.1. Then the pair (Ψ, Φ) has a well-posed common attractor-based problem.
Proof. See [1]

Theorem 4.3. Consider a complete partial metric space (Y, p) with $\Psi, \Phi : \mathcal{C}^p(Y) \to \mathcal{C}^p(Y)$ defined as in Theorem 3.7. Then the pair (Ψ, Φ) has a well-posed common attractor-based problem.
Proof. See [1]

[1] M. Khumalo, T. Nazir, V. Makhoshi, Generalized iterated function system for common attractors in partial metric spaces, AIMS Mathematics, 7(7) (2022) 13074-13103.

[1]

Application to Functional Equations

In this section, we apply our obtained results to solve a functional equation arising in the dynamic programming.
Let W_1 and W_2 be two Banach spaces with $U \subseteq W_1$ and $V \subseteq W_2$. Suppose that

$$\kappa\colon U \times V \longrightarrow U, \quad g_1, g_2\colon U \times V \longrightarrow \mathbb{R}, \quad h_1, h_2\colon U \times V \times \mathbb{R} \longrightarrow \mathbb{R}.$$

If we consider U and V as the state and decision spaces respectively, then the problem of dynamic programming reduces to the problem of solving the functional equations:

$$q_1(x) = \sup_{y \in V}\{g_1(x,y) + h_1(x,y,q_1(\kappa(x,y)))\}, \text{ for } x \in U \tag{5.1}$$

$$q_2(x) = \sup_{y \in V}\{g_1(x,y) + h_2(x,y,q_2(\kappa(x,y)))\}, \text{ for } x \in U. \tag{5.2}$$

The equations (5.1) and (5.2) can be reformulated as

$$q_1(x) = \sup_{y \in V}\{g_2(x,y) + h_1(x,y,q_1(\kappa(x,y)))\} - b, \text{ for } x \in U \tag{5.3}$$

$$q_2(x) = \sup_{y \in V}\{g_2(x,y) + h_2(x,y,q_2(\kappa(x,y)))\} - b, \text{ for } x \in U, \tag{5.4}$$

where $b > 0$.

We study the existence and uniqueness of the bounded solution of the functional equations (5.3) and (5.4) arising in dynamic programming in the setup of partial metric spaces.
Let $B(U)$ denote the set of all bounded real valued functions on U. For an arbitrary $\eta \in B(U)$, define $\|\eta\| = \sup_{t \in U} |\eta(t)|$. Then $(B(U), \|\cdot\|)$ is a Banach space. Now consider

$$p_{B}(\eta, \xi) = \sup_{t \in U} |\eta(t) - \xi(t)| + b,$$

where $\eta, \xi \in B(U)$. Then p_B is a partial metric on $B(U)$ (see also[1]).

[1] M. Abbas and B. Ali, Fixed point of Suzuki-Zamfirescu hybrid contractions in partial metric spaces via partial Hausdorff metric, Fixed Point Theory Appl., Article 21 (2013), 16 pages.

Assume that:

(D_1) : g_1, g_2, h_1 and h_2 are bounded and continuous.

(D_2) : For $x \in U$, $\eta \in B(U)$ and $b > 0$, take $\Psi, \Phi : B(U) \to B(U)$ as

$$\Psi\eta(x) = \sup_{y \in V}\{g_2(x,y) + h_1(x,y,\eta(\kappa(x,y)))\} - b, \text{ for } x \in U, \quad (5.5)$$

$$\Phi\eta(x) = \sup_{y \in V}\{g_2(x,y) + h_2(x,y,\eta(\kappa(x,y)))\} - b, \text{ for } x \in U. \quad (5.6)$$

Moreover, for every $(x, y) \in U \times V$, $\eta, \xi \in B(U)$ and $t \in U$ implies

$$|h_1(x, y, \eta(t)) - h_2(x, y, \xi(t))| \leq \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)) - 2b, \quad (5.7)$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\eta(t), \xi(t)) &= \max\{p_B(\eta(t), \xi(t)), p_B(\eta(t), \Psi\eta(t)), \\ &\quad p_B(\xi(t), \Phi\xi(t)), \\ &\quad \frac{p_B(\eta(t), \Phi\xi(t)) + p_B(\xi(t), \Psi\eta(t))}{2}\Big\}. \end{aligned}$$

Theorem 5.1. Assume that the conditions (D_1) and (D_2) hold. Then, the functional equations (5.3) and (5.4) have a unique common and bounded solution in $B(U)$.

Proof. Note that $(B(U), p_B)$ is a complete partial metric space. By (D_1), Ψ and Φ are self-mappings of $B(U)$. By (5.5) and (5.6) in (D_2), it follows that for any $\eta, \xi \in B\,(U)$ and $b > 0$, choose $x \in U$ and $y_1, y_2 \in V$ such that

$$\Psi\eta < g_2(x, y_1) + h_1(x, y_1, \eta(\kappa(x, y_1))) \tag{5.8}$$

$$\Phi\xi < g_2(x, y_2) + h_2(x, y_2, \xi(\kappa(x, y_2))), \tag{5.9}$$

which further implies that

$$\Psi\eta \geq g_2(x, y_2) + h_1(x, y_2, \eta(\kappa(x, y_2))) - b \tag{5.10}$$
$$\Phi\xi \geq g_2(x, y_1) + h_2(x, y_1, \xi(\kappa(x, y_1))) - b. \tag{5.11}$$

From (5.8) and (5.11) together with (5.7) implies

$$\begin{aligned}\Psi\eta(t) - \Phi\xi(t) &< h_1(x, y_1, \eta(\kappa(x, y_1))) - h_2(x, y_1, \xi(\kappa(x, y_1))) + b \\ &\leq |h_1(x, y_1, \eta(\kappa(x, y_1))) - h_2(x, y_1, \xi(\kappa(x, y_1)))| + b \\ &\leq \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)) - b. \qquad (5.12)\end{aligned}$$

From (5.9) and (5.10) together with (5.7) implies

$$\begin{aligned}\Phi\xi(t) - \Psi\eta(t) &< h_2(x, y_2, \xi(\kappa(x, y_2))) - h_1(x, y_2, \eta(\kappa(x, y_2))) + b \\ &\leq |h_1(x, y_2, \eta(\kappa(x, y_2))) - h_2(x, y_2, \xi(\kappa(x, y_2)))| + b \\ &\leq \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)) - b. \qquad (5.13)\end{aligned}$$

From (5.12) and (5.13), we get

$$|\Psi\eta(t) - \Phi\xi(t)| + b \leq \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)). \tag{5.14}$$

The inequality (5.14) implies that

$$p_B(\Psi\eta(t), \Phi\xi(t)) \leq \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)), \tag{5.15}$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\eta(t), \xi(t)) \quad = \quad & \max\{p_B(\eta(t), \xi(t)), p_B(\eta(t), \Psi\eta(t)), \\ & p_B(\xi(t), \Phi\xi(t)), \\ & \left.\frac{p_B(\eta(t), \Phi\xi(t)) + p_B(\xi(t), \Psi\eta(t))}{2}\right\}. \end{aligned}$$

Therefore, all conditions of Corollary 3.4 hold. Thus, there exists a common fixed point of Ψ and Φ, that is, $\eta^* \in B(U)$, where $\eta^*(t)$ is a common solution of functional equations (5.3) and (5.4).

Conclusion

The results of this presentation broadened the scope of iterated function system and fixed point theory of a pair of mappings by incorporating the generalized contraction approaches. We obtained unique common attractors with the assistance of finite families of generalized contractive mappings, that belong to the special class of mappings defined on a partial metric space. The well-posedness of these obtained results is also established.

The ideas in this work, being discussed in the setting of partial metric spaces, are completely fundamental. Hence, they can be made better, when presented in the extended generalized metric spaces, like dislocated metric, semi metric, b-metric spaces, G-metric spaces and some other pseudo-metric or quasi metric spaces.

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I Thank You