Numerical approximation to a nonlinear equation with global differentiation and intregral operator

Hangwelani MAGAU

Central University of Technology, Welkom Campus,South Africa, 9595

65TH SAMS CONGRESS
Department of Mathematics, STELLENBOSCH UNIVERSITY
Stellenbosch Central, Stellenbosch, South Africa

06-08 December, 2022

Outline

- Riemann-Stieltjies integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Outline

- Riemann-Stieltjies integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Outline

- Riemann-Stieltjies integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Outline

- Riemann-Stieltjies integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Outline

- Riemann-Stieltjes integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Outline

- Riemann-Stieltjies integral
- The global derivative
- Numerical schemes with global darivative
- Mid-point method
- Caputo mid-point numerical scheme
- Consistancy of the method

Riemann-Stieltjes

Let α be a monotone increasing function on [a,b] and f be a bounded real-valued function on [a,b]. For each $\rho = \{x_0, x_1, ...x_n\}$ of [a,b] set $\Delta\alpha_i = \alpha x_i - \alpha x_{i-1}, i = 1, 2, 3....n$. Since Δ is monotone increasing , $\Delta\alpha_i \geq 0$, $\forall i$. Let

$$m_i = inf\{f(t) : t \in [x_{i-1}, x_i]\}$$

$$M_i = sup\{f(t) : t \in [x_{i-1}, x_i]\}$$

The upper Riemann-Stieltjes sum of f with respect to α and the partition ρ, is defined by

$$U(\rho, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

The Riemann-Stieltjes sum of f with respect to α and the partition ρ, is defined by

$$L(\rho, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

Riemann-Stieltjes

Let ρ be any partition of [a,b], then the following sum can be found.

$$\sum_{i=1}^{n} \Delta\alpha_i = [\,(\alpha(x_1) - \alpha(x_0)) + (\alpha(x_2) - \alpha(x_2)) + ... + (\alpha(x_n) - \alpha(x_{n-1}))]$$
$$= (\alpha(x_n) - \alpha(x_0))$$
$$= \alpha(b) - \alpha(a)$$

Since $M_i \leq M \forall i$ and $\Delta\alpha_i \geq 0$

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M \Delta\alpha_i$$
$$= M \sum_{i=1}^{n} \Delta\alpha_i$$
$$= M(\alpha(b) - \alpha(a))$$

Therefore $U(\rho, f, \alpha) \leq M(\alpha(b) - \alpha(a))$ and $L(\rho, f, \alpha) \geq m(\alpha(b) - \alpha(a))$

Riemann-Stieltjes

The Riemann-Stieltjes integral of a real-valued function f of a real variable on the interval $[a, b]$ with respect to another real-to-real function g is denoted by

$$\int_a^b f(x) \mathrm{d}g(x).$$

Its definition uses a sequence of partitions P of the interval $[a, b]$

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

Global derivatives

- We know that if y=f(x) then $\frac{dy}{dx}$ will be the rate of change of the variable y with respect to x. In other words, the rate of change,
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1}$$
- We can think of the above equation as a proportion of a continuous function f(x) between x_1 and x_2 and the function g(t)=t.

Definition

Let f be a continuous function and g(x) be a positive non-constant and increasing function, such that if $a < b$, then $g(a) < g(b)$. The global rate of change between a and b is given by:

$$M = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{2}$$

Global derivatives

Definition

Let f be a continuous function and g(x) be a positive continuous increasing function in the interval [a,b] and non-zero for all t $\in$ [a,b]. The derivative of f with respect to the function g is given by:

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)} \tag{3}$$

Global derivatives-Product rule

Let f_1, f_2 and g be differentiable, then.

$$\begin{aligned}
D_g f_1 f_2 &= \lim_{h\to 0} \frac{f_1(t+h)f_2(t+h) - f_1(t)f_2(t)}{g(t+h)-g(t)} \\
&= \lim_{h\to 0} \frac{f_1(t+h)f_2(t+h) - f_1(t+h)f_2(t) + f_1(t+h)f_2(t) - f_1(t)f_2(t)}{g(t+h)-g(t)} \\
&= \lim_{h\to 0} \frac{f_1(t+h)[f_2(t+h) - f_2(t)] + f_2(t)[f_1(t+h) - f_1(t)]}{g(t+h)-g(t)} \\
&= \lim_{h\to 0} \frac{f_1(t+h)[f_2(t+h) - f_2(t)]}{g(t+h)-g(t)} + \lim_{h\to 0} \frac{f_2(t)[f_1(t+h) - f_1(t)]}{g(t+h)-g(t)} \\
&= (\lim_{h\to 0} f_1(t+h)) \lim_{h\to 0} \frac{[f_2(t+h) - f_2(t)]}{g(t+h)-g(t)} + f_2(t) \lim_{h\to 0} \frac{[f_1(t+h) - f_1(t)]}{g(t+h)-g(t)} \\
D_g f_1 f_2 &= f_1(t) D_g f_2(t) + f_2(t) D_g f_1(t)
\end{aligned}$$

Global derivatives- Quotientt rule

$$
\begin{aligned}
D_g \frac{1}{f} &= \lim_{t \to t_1} \frac{\frac{1}{f(t+t_1)} - \frac{1}{f(t)}}{g(t+t_1) - g(t)} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t+t_1)}{f(t+t_1) f(t)} * \frac{1}{g(t+t_1) - g(t)} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t+t_1)}{g(t+t_1) - g(t)} * \frac{1}{f(t+t_1) f(t)} \\
&= \lim_{t \to t_1} -\frac{f(t+t_1) - f(t)}{g(t+t_1) - g(t)} * \frac{1}{f(t+t_1) f(t)} \\
D_g \frac{1}{f} &= -\frac{D_g f}{f^2(t)}
\end{aligned}
$$

Global derivative-Composition

$$
\begin{aligned}
D_g f_1(f_2(t)) &= \lim_{t\to t_1} \frac{f_1(f_2(t+t_1) - f_1(f_2(t))}{g(t+t_1) - g(t)} \\
&= \lim_{t\to t_1} \frac{f_1(f_2(t+t_1) - f_1(f_2(t))}{g(t+t_1) - g(t)} * \frac{f_2(t+t_1) - f_2(t)}{f_2(t+t_1) - f_2(t)} \\
&= \lim_{t\to t_1} \frac{f_1(f_2(t+t_1) - f_1(f_2(t))}{f_2(t+t_1) - f_2(t)} * \lim_{t\to t_1} \frac{f_2(t+t_1) - f_2(t)}{g(t+t_1) - g(t)} \\
&= \lim_{t\to t_1} \frac{f_1(f_2(t+t_1) - f_1(f_2(t))}{f_2(t+t_1) - f_2(t)} * D_g f_2 \\
&= f'(f_2(t)) * D_g f_2 \\
D_g f_1(f_2(t)) &= f'(f_2(t)) * D_g f_2
\end{aligned}
$$

Global derivatives

Without any loss of generality we assume that $(D_g f)(t)$ exist let $t > t_1$

- If $D_g f(t) = 0 \implies f(t) = f(t_1)$, Then the function f is a constant
- If $D_g f(t) > 0 \implies f(t) > f(t_1)$, Then the function f increases
- If $D_g f(t) < 0 \implies f(t) < f(t_1)$, Then the function f decreasing

Global derivatives

- When g(t)=t

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \tag{4}$$

- we then recover the classical differential operator
- To recover the fractal derivative we let $g(t) = t^\alpha$

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha}, t > 0, \alpha > 0 \tag{5}$$

Global derivatives

- $g(t) = \frac{t^{2-\alpha}}{2-\alpha}$. we get

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{2-\alpha} - t_1^{2-\alpha}}(2-\alpha), t > 0, 1 \leq \alpha < 2 \qquad (6)$$

Also if $g(t) = t^{\beta(t)}$

$$D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{t^{\beta(t)}} - t_1^{t^{\beta(t)}}} \qquad (7)$$

Global derivatives

- For g(t)=t we get

$$D_t f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} = f'(t_1)$$

- For $g(t) = t^\alpha$ we get

$$\begin{aligned} D_{t^\alpha} f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^\alpha - t_1^\alpha} \times \frac{t - t_1}{t - t_1} \\ &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{t^\alpha - t_1^\alpha} \\ &= f'(t_1) \times \frac{t_1^{1-\alpha}}{\alpha} \end{aligned}$$

Global derivatives

- For $g(t) = \frac{t^{2-\alpha}}{2-\alpha}$ we get

$$\begin{aligned}
D_g f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{\frac{t^{2-\alpha}}{2-\alpha} - \frac{t_1^{2-\alpha}}{2-\alpha}} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{\frac{t^{2-\alpha}}{2-\alpha} - \frac{t_1^{2-\alpha}}{2-\alpha}} \times \frac{t - t_1}{t - t_1} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{\frac{t^{2-\alpha}}{2-\alpha} - \frac{t_1^{2-\alpha}}{2-\alpha}} \\
&= f'(t_1) \times t_1^{\alpha-1}
\end{aligned}$$

Global derivatives

- If $g(t) = t^{B(t)}$ we get

$$
\begin{aligned}
D_g f(t_1) &= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{B(t)} - t_1^{B(t_1)}} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t^{B(t)} - t_1^{B(t_1)}} \times \frac{t - t_1}{t - t_1} \\
&= \lim_{t \to t_1} \frac{f(t) - f(t_1)}{t - t_1} \times \frac{t - t_1}{t^{B(t)} - t_1^{B(t_1)}} \\
&= f'(t_1) \times \frac{t^{-B(t_1)}}{B'(t_1)\ln(t_1) + \frac{B(t_1)}{t_1}}
\end{aligned}
$$

If f and g are differential ,then

$$
D_g f(t_1) = \lim_{t \to t_1} \frac{f(t) - f(t_1)}{g(t) - g(t_1)} = \frac{f'(t_1)}{g'(t_1)} \tag{8}
$$

Convolution

Definition

The convolution of two positive functions g(t)and f(t) is a function of t, denotd by (f*g)(t), defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{9}$$

Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ *and* $\mathscr{L}\{g(t)\} = G(s)$*, then*

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s) \tag{10}$$

Proof.

By the definition of convulution:

$$
\begin{aligned}
&\mathscr{L}\{\int_0^\infty f(\tau)g(t-\tau)dt\} \\
&= \int_0^\infty \big(e^{-st}\int_0^\infty f(\tau)g(t-\tau)d\tau\big)dt \\
&= \int_0^\infty \big(f(\tau)\int_0^\infty e^{-st}g(t-\tau)dt\big)d\tau \\
u &= t-\tau, then \\
&= \int_0^\infty \big(f(\tau)\int_0^\infty e^{-s(u+\tau)}g(u)du\big)d\tau \\
&= \int_0^\infty \big(e^{-\tau s}f(\tau)\int_0^\infty e^{-su}g(u)du\big)d\tau \\
&= \int_0^\infty \big(e^{-\tau s}f(\tau)G(u)\big)d\tau \\
&= G(u)\int_0^\infty \big(e^{-\tau s}f(\tau)\big)d\tau \\
&= F(s)G(s)
\end{aligned}
$$

□

Extention to Non-local operators

- Nowadays the concept non-local operators is widely used in the field of mathematics. With that being said it calls for those operators to be timiously visted to improve and modify them for a better understanding of the changing world around us.

Abdon Atangana and Baleanu [25] proposed the following definitions.

Definition

Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$, the Atangana-Baleanu fractional derivative in Caputo sense is:

$$ {}_{a}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\alpha}\left[-\alpha \frac{(t-\tau)^{\alpha}}{1-\alpha}\right] f'(\tau) d\tau \qquad (11) $$

Extention to Non-local operators

Definition

Let $f \in H^1(a1, a2), a2 > a1, \alpha \in [0, 1]$ not necessary differentiable, the Atangana-Baleanu fractional derivative in Reimman-Liouville sense is given as:

$$ {}_{a}^{ABR}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] f(\tau) d\tau \qquad (12) $$

Definition

Suppose the function does not belong to $H^1(a, b)$.
Caputo-Fabrizio derivative of fractional order α is defined as:

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) exp \left[-\alpha \frac{t-\tau}{1-\alpha} \right] d\tau \qquad (13) $$

Again, $M(\alpha)$ is such that $M(0) = M(1) = 1$

Extention to Non-local operators

Definition

Let f(t) be a continuoius function and g(t) be a non-constant increasing positive function. Let $K(t)$ be a Kernel singular or non-singular for $0 < \alpha \leq 1$.A fractional global derivative of Caputo sence is given by.

$$ {}_0^C D_g^\alpha f(t) = D_g f(t) * K(t) $$

derivative of Caputo sence is given by.

$$ {}_0^{RL} D_g^\alpha f(t) = D_g(f(t) * K(t)) $$

- Here, * denotes the convolution operator. Now,If the Kernel $K(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ We then have the power-law type.

$$ 0^{RL} D_g^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} D_g \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau \qquad (14) $$

Extention to Non-local operators

And with Caputo we get.

$$0^{C}D_g^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t D_g f(\tau)(t-\tau)^{-\alpha}d\tau \qquad (15)$$

To recover the Caputo-Fabrizio we take $K(t) = \frac{exp[-\frac{\alpha}{1-\alpha}t]}{1-\alpha}$

$$0^{CF}D_g^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t D_g f(\tau)exp[-\frac{\alpha}{1-\alpha}(t-\tau)]d\tau \qquad (16)$$

$$0^{RC}D_g^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}D_g\int_0^t f(\tau)exp[-\frac{\alpha}{1-\alpha}(t-\tau)]d\tau \qquad (17)$$

Extention to Non-local operators

We recover the Atangana-Baleanu derivative for $K(t) = \frac{AB(\alpha)}{1-\alpha} E_\alpha[-\frac{\alpha}{1-\alpha} t^\alpha]$

$$0^{ABC} D_g^\alpha f(t) = \frac{AB\alpha)}{1-\alpha} \int_0^t D_g f(\tau) E_\alpha[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha] d\tau \quad (18)$$

$$0^{ABR} D_g^\alpha f(t) = \frac{AB(\alpha)}{1-\alpha} D_g \int_0^t f(\tau) E_\alpha[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha] d\tau \quad (19)$$

Consider the following Cauchy problem

$$D_g y(t) = f(t, x(t)), y(0) = x_0 \tag{20}$$

We assume that $g(t)$ and $y(t)$ are differentiable, then

$$\frac{d}{dt} y(t) = g'(t) f(t, y(t)) \tag{21}$$

Adams-Bashforth numerical scheme with global derivative

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} g'(t) f(\tau, y(\tau)) d\tau \tag{22}$$

Making $\Theta(t, y(t)) = g'(t) f(t, y(t))$ and letting P be the interpolating function, with

$$P(\tau) = \Theta(t_n, y_n) \frac{\tau - t_{n-1}}{t_n - t_{n-1}} - \Theta(t_{n-1}, y_{n-1}) \frac{\tau - t_n}{t_n - t_{n-1}} \tag{23}$$

$$y_{n+1} = y_n + \frac{3}{2} \Delta t \Theta(t_n, y_n) - \frac{1}{2} \Delta t \Theta(t_{n-1}, y_{n-1}) \tag{24}$$

Replacing $\Theta(t_n, y_n) = g'(t_n) f(t_n, y_n)$

$$y_{n+1} = y_n + \frac{3}{2} \Delta t \frac{g(t_{n+1}) - g(t_n)}{\Delta t} f(t_n, y_n) - \frac{1}{2} \Delta t \frac{g(t_n) - g(t_{n-1})}{\Delta t} f(t_{n-1}, y_{n-1}) \tag{25}$$

Finally

$$y_{n+1} = y_n + \frac{3}{2} \Delta t (g(t_{n+1}) - g(t_n)) f(t_n, y_n) - \frac{1}{2} \Delta t (g(t_n) - g(t_{n-1})) f(t_{n-1}, y_{n-1}) \tag{26}$$

Linear polynomials numerical scheme with global derivative

Let

$$D_g y(t) = y(t, y(t)), y(0) = y_0$$

$$y(t) = y(0) + \int_0^t f(\tau, y(\tau)) g'(\tau) d\tau$$

at $t = t_{n+1}$

$$y(t_{n+1}) = y(0) + \int_0^{t_{n+1}} f(\tau, y(\tau)) g'(\tau) d\tau$$

$$y(t_{n+1}) - y(0) = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f(\tau, y(\tau)) \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} d\tau$$

We now approximate $f(\tau, y(\tau))$ by a polynomial $P_j(\tau)$ within the interval $[t_j, t_{j+1}]$

$$P_j \tau = f(t_j, y_j) + \frac{f(t_{j+1}, y_{j+1}) - f(t_j, y_j)}{h} (\tau - t_j)$$

Then

Linear polynomials numerical scheme with global derivative

$$y(t_{n+1}) - y(t_n) = \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} P_j(\tau, y(\tau)) d\tau$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \int_{t_y}^{t_{j+1}} f(t_j, y_j) + \frac{f(t_{j+1}, y_{j+1}) - f(t_j, y_j)}{h} (\tau - t_j) d\tau$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \left[h f(t_j, y_j) + \frac{f(t_{j+1}, y_{j+1}) - f(t_j, y_j)}{h} \int_{t_j}^{t_{j+1}} (\tau - t_j) d\tau \right]$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \left[h f(t_j, y_j) + \frac{f(t_{j+1}, y_{j+1}) - f(t_j, y_j)}{h} \left. \frac{(\tau - t_j)^2}{2} \right|_{t_j}^{t_{j+1}} \right]$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} h \left[\frac{f(t_{j+1}, y(t_{j+1})) + f(t_j, y(t_j))}{2} \right]$$

The approximation scheme is:

$$y(t_{n+1}) = y(t_n) + \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} h \left[\frac{f(t_{j+1}, y(t_{j+1})) + f(t_j, y(t_j))}{2} \right] \quad (27)$$

Lagrange polynomials numerical scheme with global derivative

Again

$$D_g y(t) = y(t, y(t)), y(0) = y_0$$

$$y(t) = y(0) + \int_0^t f(\tau, y(\tau)) g'(\tau) d\tau$$

at $t = t_{n+1}$

$$y(t_{n+1}) = y(0) + \int_0^{t_{n+1}} f(\tau, y(\tau)) g'(\tau) d\tau$$

$$y(t_{n+1}) - y(t_n) = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f(\tau, y(\tau)) \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} d\tau$$

We now approximate $f(\tau, y(\tau))$ by a polynomial $P_j(\tau)$ within the interval $[t_j, t_{j+1}]$

$$P_j \tau = \frac{\tau - t_{j-1}}{h} f(t_j, y_j) + \frac{\tau - t_j}{h} f(t_{j-1}, y_{j-1})$$

Then

Linear polynomials numerical scheme with global derivative

$$y(t_{n+1}) - y(n) = \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} P_j(\tau, y(\tau))d\tau$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \int_{t_y}^{t_{j+1}} \frac{\tau - t_{j-1}}{h} f(t_j, y_j) + \frac{\tau - t_j}{h} f(t_{j-1}, y_{j-1}) d\tau$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \frac{1}{h} \left[f(t_j, y_j) \int_{t_j}^{t_{j+1}} (\tau - t_{j-1}) d\tau - f(t_{j-1}, y_{j-1}) \int_{t_j}^{t_{j+1}} (\tau - t_j) d\tau \right]$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \frac{1}{h} \left[f(t_j, y_j) \left. \frac{(\tau - t_{j-1})^2}{2} \right|_{t_j}^{t_{j+1}} (\tau - t_{j-1}) d\tau - f(t_{j-1}, y_{j-1}) \frac{(\tau - t_j}{2}\right.$$

$$= \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \frac{h}{2} \left[3f(t_j, y(t_j)) - f(t_{j-1}, y_{j-1}) \right]$$

The approximation scheme is:

$$y(t_{n+1}) = y(n) + \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} \frac{h}{2} \left[3f(t_j, y(t_j)) - f(t_{j-1}, y_{j-1}) \right]$$

Error analysis for Linear polynomials numerical scheme with global derivative

Theorem

Let

$$y(t_{n+1}) = y(n) + \sum_{j=0}^{n} \frac{g(t_{t_{j+1}}) - g(t_j)}{\Delta t} h \Big[\frac{f(t_{j+1}, y(t_{j+1})) + f(t_j, y(t_j))}{2} \Big] + R \tag{29}$$

Where:

$$|R| \leq \frac{h^2}{12} max|f''| \sum_{j=0}^{n} \Big[g(t_{j+1}) - g(t_j) \Big] \Big[-6j^2 + 6j + 5 \Big] \tag{30}$$

Global numerical scheme using Mid-point method with Caputo Fractional Darivative

For the nonlinear fractional order equation:

$$\begin{cases} cD_g^\alpha y(t) = f(t, y(t)) \text{ if } t \geqslant 0 \\ y(0) = y_0 \end{cases}$$

$$\begin{cases} cD_t^\alpha y(t) = g'(t) f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$$\begin{cases} y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau) f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau \\ y(0) = y_0 \end{cases}$$

$$\begin{cases} \text{at } t_{n+1} = t \\ y(t_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} g'(\tau) f(\tau, y(\tau)) (t_{n+1} - \tau) d\tau \\ y(0) = y_0 \end{cases}$$

Global numerical scheme using Mid-point method with Caputo Fractional Darivative

$$y\left(t_{n+1}\right)=y(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} g'(\tau)f(\tau,y(\tau))\left(t_{n+1}-\tau\right)^{\alpha-1}d\tau$$

$$=y(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{g\left(t_{j+1}\right)-g\left(t_j\right)}{\Delta}f\left(\frac{t_j+t_{j+1}}{2},\frac{y_j+y_{j+1}}{2}\right)$$

$$\times\left(t_{n+1}-\tau\right)d\tau$$

$$=y(0)+\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)}\sum_{j=0}^{n}\left(g\left(t_{i+1}\right)-g\left(t_j\right)\right)$$

$$\times f\left(t_i+\frac{h}{2},\frac{y_j+y_{j+1}}{2}\right)\left\{(n-j+1)^{\alpha}-(n-j)^{\alpha}\right\}$$

Global numerical scheme using Mid-point method with Caputo Fractional Darivative

$$= y(0) + \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \left(g\left(t_{i+1}\right) - g(t,)\right)$$

$$\times f\left(t_i + \frac{h}{2}, \frac{y_i + y_{j+1}}{2}\right) \left\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\right\}$$

$$+ \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \left(g\left(t_{n+1}\right) - g\left(t_n\right)\right)$$

$$\times f\left(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}^{p}}{2}\right) \left\{(n-j+1)^{\alpha} - (n_{-j})^{\alpha}\right\}$$

Where

$$y_{n+1}^{p} = \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \left(g\left(t_{i+1}\right) - g\left(t_j\right)\right) \times f\left(t_i, y_j\right)$$

$$\left[(n-j+1)^{\alpha} - (n-j)^{\alpha}\right]$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

We wish to show that

$$\lim_{h \to 0} |y(n+1) - y_{n+1}| = 0$$

Exact Solution is of this form:

$$y\left(t_{n+1}\right) = y(t_n) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} \left(t_{n+1} - \tau\right)^{\alpha - 1} f(\tau, y(\tau)) g'(\tau) d\tau$$

our numerical Solution is:

$$y_{n+1} = y_n + \frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \left[g\left(t_{j+1}\right) - g\left(t_j\right)\right] f\left(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\right)$$

$$\times \left[(n-j+1)^{\alpha} - (n-j)^{\alpha}\right]$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

Subtracting the two equation, we get

$$
\begin{aligned}
& y\left(t_{n+1}\right)-y_{n+1} \\
= & \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}}\left(t_{n+1}-\tau\right)^{\alpha-1} f(\tau, y(\tau)) g^{\prime}(\tau) d \tau \\
& -\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^n\left(g\left(t_{j+1}\right)-g\left(t_j\right)\right) f\left(t_j+\frac{h}{2}, \frac{y_j+y_{j+1}}{2}\right) \\
& \times\left[(n-j+1)^\alpha-(n-j)^\alpha\right] \\
& =\frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} g^{\prime}(\tau) f(\tau, y(\tau))\left(t_{n+1}-\tau\right)^{\alpha-1} d \tau-\frac{1}{\Gamma(\alpha)} \sum_{v=0}^n \\
& \int_{t_j}^{t_{n+1}} g^{\prime}(\tau) f\left(t_j+\frac{h}{2}, \frac{y_i+y_{j+1}}{2}\right)\left(t_{n+1}-\tau\right)^{\alpha-1} d \tau
\end{aligned}
$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

$$= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(\tau)\,(t_{n+1} - \tau)^{\alpha-1}$$

$$\times \left[f(\tau, y(\tau)) - f\left(t + \frac{h}{2}, \frac{y_j + y_{i+1}}{2} \right) \right] d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{t+1}} \left| g'(\tau) \right| \left| f(\tau, y(\tau)) - f\left(t + \frac{h}{2}, \frac{y_j + y_{j+1}}{2} \right) \right|$$

$$\times (t_{n+1} - \tau)^{\alpha-1}\, d\tau$$

Since $f(t, y(t)$ is differentiable for all t, we are very much greatfull to the Mean-Vaiue Theorem. Thus $\exists$ C $\in$ (Such that we can find $f'(c)$, such that

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \sup_{u(t_j, t_+)} \left| g'(t) \right| f'(c, y(c)) \int_{t_j}^{t_j+1} \left(\tau - \left(t_j + \frac{h}{2} \right) \right)$$

$$\times \left(t_{n+1} - \tau \right)^{\alpha - 1} d\tau$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

Lets Shift our focus to the following integral

$$\int_{t_j}^{t_j+1} \left(\tau - \left(t, + \frac{h}{2}\right)\right) (t_{n+i} - \tau)^{\alpha-1}\, d\tau \tag{31}$$

$$\int_{t_j}^{t_{j+1}} \tau\, (t_{n+1} - \tau)^{\alpha-1}\, d\tau$$

$$= \int_{0}^{t_{j+1}} \tau\, (t_{n+1}\tau)^{\alpha-1}\, d\tau - \int_{0}^{t_j} \tau\, (t_{n+1} - \tau)^{\alpha-1}\, d\tau$$

$$= \int_{0}^{\frac{t_{n+1}}{t_{n+1}}} u t_{n+1}\, (t_{n+1} - t_{n+1}u)^{\alpha-1}\, du -$$

$$\int_{0}^{\frac{t_j}{t_{n+1}}} t_{n+1}u\, (t_{n+1} - t_{n+1}u)\, t_{n+1}^{\alpha-1} d\tau$$

$$= t_{n+1}^{\alpha+1} \int_{0}^{\frac{t_{j+1}}{t_{n+1}}} u^{2-1}(1-u)^{\alpha-1} du - t_{n+1}^{\alpha+1} \int_{0}^{\frac{t_n}{t_{n+1}}} u^{2-1}(1-u)^{\alpha-1} du$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

$$= t_{n+1}^{\alpha+1}\left[B\left(\frac{t_{s+1}}{t_{n+1}},2,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},2,\alpha\right)\right] \tag{32}$$

Here we make use of the incomplete beta function

$$B(x,a,b)=\int_0^x t^{\alpha-1}(1-t)^{b-1}dt \tag{33}$$

now:

$$\int_{t_j}^{t_{j+1}}\left(t_j+\frac{h}{2}\right)(t_{n+1}-\tau)^{\alpha-1}\,d\tau \text{ is easy to compute}$$

$$\int_{t_j}^{t_{j+1}}\left(t_j+\frac{h}{2}\right)(t_{n+1}-\tau)^{\alpha-1}\,d\tau =$$

$$\left(t_j+\frac{h}{2}\right)\left[\frac{(t_{n+1}-t_j)}{\alpha}\right)^{\alpha}-\frac{(t_{n+1}-t_{j+1})^{\alpha}}{\alpha}$$

Consistancy of the Global Mid-point method with Caputo Fractional Darivative

Putting everything together

$$B\left(\frac{t_{j+1}}{t_{n+1}},2,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},2,\alpha\right)\Bigg]$$

$$-\,h^{\alpha+1}\left(j+\frac{1}{2}\right)\left[\frac{(n-j+1)^{\alpha}-(n-j)^{\alpha}}{\alpha}\right]$$

$$\leq \frac{\|g'\|_{\infty}}{\Gamma(\alpha)}\left|f'(c,y(c))\right|h^{\alpha+1}\sum_{j=0}^{n}\left\{(n+1)^{\alpha+1}\left[B\left(\frac{t_{i+1}}{t_{n+1}},2,\alpha\right)\right.\right.$$

$$\left.\left.-B\left(\frac{t_j}{(t_{n+1}},2,\alpha\right)-\frac{1}{\alpha}\left(j+\frac{1}{2}\right)\left[\frac{(n-j+1)^{\alpha}-(n-j)^{\alpha}}{\alpha}\right]\right]\right\}$$

finally it is clear that

$$\lim_{h\to 0}\left|y\left(t_{n+1}\right)-y_{n+1}\right|=0$$

Bibliography

A. Atangana and D. Baleanu,*New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model thermal science* Thermal Science, 316 (2016)arXiv preprint arXiv:1602.03408, 2016

A. Atangana, *Extension of rate of change concept: From local to nonlocal operators with applications*, Results in physics 2010.DOI 10.1140/epjp/i2018-11863-9

B. Karaagae, K.M. Owolabi and K.S Nisar,*Analysis and Dynamics of Illicit Drug Use Described by Fractional Derivative with Mittag-Leffler Kernel*, (Computers, Materials and Continua, 2020).

Bibliography

- J. Abate and P.P. Valko, Multi-precision Laplace transform inversion, *International Journal for Numerical Methods in Engineering* **60 (5)** (2004) 979–993.
- M.Caputo and M.Frabizio,*A new Definition of Fractional Derivative without Singular Kernel*, (Progress in Fractional Differentiation and Applications, 2015).
- S.H.Strogatz,*Nonlinear dynamics and chaos: application to physics, Biology, Chemistry, and Engineering*, (Addison-Wesley Publishing company(1994)).130-144
- A.Atangana and G.F Gomez,*Decolonization of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena*, (The European Physical journal plus 133:166.166 (2018)
- M. Toufik and A.Atangana,*New numerical approximation of fractional derivative with non-local and non-singular kernel:*