Elementary Proof of Semilattice Duality

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List of slides

1. Introduction

Some categories:

JSL join-semilattices with 0 and 0-$\vee$-preserving maps.

CZJSL compact 0-dimensional topological JSLs and continuous JSL-maps

LCZJSL locally compact 0-dimensional topological JSLs and continuous JSL-maps

The duality theorem for join-semilattices traces back to

- C W Austin, Trans. Am. Math. Soc. 109(1963), 245-256).

It was studied in detail by

- K H Hoffman, M Mislove & A Stralka (Springer LNM 396 (1974)).

Theorem (AHMS Duality) The following hom-functors, suitably enriched,*

$$\mathsf{JSL}(\_,2) : \mathsf{JSL} \to \mathsf{CZJSL} \quad \text{and} \quad \mathsf{CZJSL}(\_,2) : \mathsf{CZJSL} \to \mathsf{JSL}$$

provide a dual equivalence of categories.

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* $\mathsf{JSL}(L,2)$ inherits a CZJSL structure from 2^L, and $\mathsf{CZJSL}(Z,2)$ inherits a JSL structure from 2^Z. The enriched structures will be denoted $\widehat{L}$ and $\widehat{Z}$, respectively.

2. Introduction (continued)

To prove AHMS duality, we must show:

- if L is discrete (resp., compact-zero-dimensional), then $\widehat{L}$ is compact-zero-dimensional (resp., discrete), and
- in both cases, L is naturally isomorphic with $\widehat{\widehat{L}}$.

The most difficult step lies in proving:

There are enough CZJSL characters, i.e., the CZJSL-characters of any CZJSL-object separate its elements.

Hoffman-Mislove-Stralka obtained this as a corollary of Numakura's Theorem (1957) that every CZJSL-object is a projective limit of finite semilattices. The idea of using projective limits to establish dualities is elaborated in Johnstone *Stone Spaces*, Chapter VI, where a proof of AHMS-duality using this strategy may be found.

No simple direct proof seems to exist in any published source. The proof we will give is based on a suggestion by Jimmie Lawson (August 2022). *Remarkably, it works for all locally-compact join semilattices,* but whether or not this supports a generalization of AHMS-duality is open.

3. Locally Compact 0-Dimensional Join-Semilattices

Definition. Suppose L is a LCZJSL-object.

- A continuous 0-$\vee$-morphism $\alpha : L \to 2$ is called a *character of L*.
- The set of all characters of L, endowed with the compact-open topology,* is denoted by $\widehat{L}$.

We shall use Greek letters $\alpha, \beta, \ldots$ to denote characters of L.

Important Fact. A function $\phi : L \to 2$ is a character if and only if: $\phi^{-1}(0)$ is a clopen ideal of L. (An ideal is a $\vee$-closed downset.)

Fact. $\widehat{L}$ has a natural *JSL*-structure.

Proof. The constant function 0 is a character, and if α, β are characters, so is $\alpha \vee \beta$:

- $(\alpha \vee \beta)(a \vee b) = (\alpha \vee \beta)(a) \vee (\alpha \vee \beta)(b)$ (uses commutativity of $\vee$);
- $(\alpha \vee \beta)^{-1}(0) = \{ a \in L \mid \alpha(a) = 0 \text{ and } \beta(a) = 0 \} = \alpha^{-1}(0) \cap \beta^{-1}(0)$ is clopen. □

* This is the right topology for JSL-CZJSL duality, but it is not known if it is the right choice for a generalization encompassing all locally compact join-semilattices.

4. Essential Facts about Topological Semilattices (Top-Facts)

Suppose Z is a topological semilattice.

For fixed $a \in Z$, let $a\vee$ denote the function: $a\vee : Z \to Z; z \mapsto a \vee z$. If Z is a topological semilattice, $a\vee$ is continuous. Hence for any open $U \subseteq Z$, $(a\vee)^{-1}U = \{\, z \in Z \mid a \vee z \in U \,\}$ is open.

Top-Facts:*

(i) If $U \subseteq Z$ is open, then $\downarrow U$ is open.

Proof. $\downarrow U = \{\, z \in Z \mid \text{for some } u \in U,\ u \vee z \in U \,\}$ is open, since it is the union of the open sets $(u\vee)^{-1}U$, $(u \in U)$.

(ii) If Z is T_1 and $a \in Z$, then $\downarrow a$ is closed.

Proof. $\downarrow a = \{\, z \in Z \mid a \vee z = a \,\}$ is the inverse image under $a\vee$ of the closed set $\{a\}$.

(iii) If Z is Hausdorff and $a \in Z$, then $\uparrow a$ is closed.

Proof. $a\vee$ is a continuous retraction onto $\uparrow a = a \vee Z$, and the image of a continuous retraction of a Hausdorff space is closed.

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* Cf. Proposition VI.1.13 of *Continuous Lattices and Domains.* (See bibliography)

5. Lawson's Lemma

Lawson's Lemma. Suppose L is an LCZJSL-object and $W \subseteq L$ is an open downset. Then for each $w \in W$ there is a $z_w \in W$ such that $z \leqslant z_w$ and $\downarrow z_w$ is clopen.

Proof. By local-compactness and 0-dimensionality, find a compact clopen V such that $w \in V \subseteq W$.

- Let $w < x < y < \cdots$ be a maximal (i.e., not extendable) ascending chain in V starting at w. By Top-Fact (iii), the sets $\uparrow w \supset \uparrow x \supset \uparrow y \supset \ldots$ form a nested family of closed sets, each having non-empty intersection with V. By compactness of V, there is at least one point of V in all these upsets. By maximality of the chain, this point is unique; call it z_w.
- By continuity of the meet operation, find open V' containing z_w and contained in V such that $\{ v_1 \vee v_2 \mid (v_1, v_2) \in V' \times V' \} \subseteq V$. Then z_w is the largest element of V', since for any $v \in V'$, $v \vee z_w \in V'$, and by maximality of the chain, this element cannot be strictly greater than z_w.
- It follows from Top-Facts (i) and (ii) that $\downarrow z_w = \downarrow V'$ is clopen. □

6. Useful Corollaries

Corollary. For any $a, b \in Z$, if $a \not\leq b$, then there is a clopen principal ideal that contains b and not a.

Proof. By Top-Fact (iii), the complement of $\uparrow a$ is an open downset. By hypothesis, it contains b, hence by the theorem, b is contained in a clopen principal ideal that does not contain a. □

Corollary. Every LCZJSL-object L admits a continuous, 0-$\vee$-preserving embedding in $2^{\hat{L}}$.

Proof. Define

$$a \mapsto f_a : L \to 2^{\hat{L}}$$

by $f_a(\alpha) := \alpha(a)$. As the product of all the characters of L, this is a is a continuous 0-$\vee$-map. If $a, b \in L$ and $a \neq b$, then $f_a \neq f_b$, because by Lawson's Lemma there is a character α so that $f_a(\alpha) \neq f_b(\alpha)$. □

7. Sketch of remaining steps in the proof

We must check that the definition of the topology is correct.

- **Fact.** In the compact-open topology, $\mathrm{JSL}(L,2)$ compact and zero-dimensional.

  In fact, $\mathrm{JSL}(L,2)$ is a closed sub-semilattice of 2^L. (For suppose $f \in 2^L \backslash \mathrm{JSL}(L,2)$. If $f(a) \vee f(b) \neq f(a \vee b)$ for some $a, b \in L$, then the functions that agree with f at a, b and $a \vee b$ form an open neighborhood of f that is disjoint from $\mathrm{JSL}(L,2)$.) The compact-open topology on $\mathrm{JSL}(L,2)$ is the same as that it inherits from 2^L.

- **Fact.** The compact-open topology on $\mathrm{CZJSL}(L,2)$ is discrete.

We must show that $a \mapsto f_a$ is surjective. In fact:

- **Evaluation Lemma.** Suppose L is discrete or CptZD. If $\psi : \widehat{L} \to 2$ is a character, then $\psi = \mathrm{ev}_a$ for some $a \in L$.

8. Completing the Proof

If $\theta : L \to M$, define $\widehat{\theta} : \widehat{M} \to \widehat{L}$ by $\widehat{\theta}(\beta) := \beta \circ \theta$.

Proposition As a endofunctor on JSL or on CZJSL

$$(\_) \mapsto \hat{\hat{\_}}$$

is naturally equivalent to Id.

Proof. In the following diagram, which can be read in either category, the vertical arrows are isomorphisms, and the diagram commutes, because $\hat{\hat{\theta}}$ sends evaluation at a to evaluation at $\theta(a)$:

$$\begin{array}{ccc} L & \xrightarrow{\ \theta\ } & M \\ {\scriptstyle a \mapsto f_a}\downarrow & & \downarrow{\scriptstyle a \mapsto f_a} \\ \hat{\hat{L}} & \xrightarrow{\ \hat{\hat{\theta}}\ } & \hat{\hat{M}} \end{array}$$

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