On products of supersoluble subgroups

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Dedicated to the memory of AGR Stewart

SAMS Conference
6 December, 2022

Outline

History and Motivation

Mutually permutable products

Weakly mutually *sn*-permutable products

Open Questions

History of products of finite groups

Note: All groups are finite

Theorem [Burnside, 1904] A group of which is a product of two Sylow subgroups is soluble.

Theorem [Hall, 1928-1937] A group is soluble if and only if it is the product of pairwise permutable Sylow subgroups.

Theorem [Kegel-Wielandt, 1958, 1962] A group which is a product of pairwise permutable nilpotent subgroups is soluble.

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History of products of finite groups

[Fitting] A group which is a product of normal nilpotent subgroups is nilpotent.

A group which is a product of normal supersoluble subgroups is not necessarily supersoluble.

[Baer, 1957] Let $G = AB$ be a product of normal supersoluble subgroups A and B and let G' be nilpotent. Then G is supersoluble.

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Mutually permutable products

Question. Can we weaken the condition of normality on A and B?

Definition. A group $G = AB$ is called a mutually permutable product of A and B if

A permutes with every subgroup of B and B permutes with every subgroup of A.

Theorem. [Asaad, Shaalan, 1989] If a group $G = AB$ is the mutually permutable product of supersoluble subgroups A and B and G' is nilpotent, then G is supersoluble.

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Theorem. A group is nilpotent if and only if every subgroup is subnormal.

Definition. A group $G = AB$ is called a mutually *sn*-permutable product of A and B if

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Theorem. [Alejandre, Ballester-Bolinches, Cossey, Pedraza-Aguilera, 2004] If a group $G = AB$ is the mutually *sn*-permutable product of supersoluble subgroups A and B and G' is nilpotent, then G is supersoluble.

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Definition. A group $G = AB$ is called a weakly mutually *sn*-permutable product of A and B if

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Weakly mutually *sn*-permutable products

Example. Let $G = S_4$ be the symmetric group of degree 4. Consider a maximal subgroup A of G which is isomorphic to S_3, and $B = A_4$, the alternating group of degree 4. Then $G = AB$ is the weakly mutually *sn*-permutable product of A and B. However, G is not a mutually *sn*-permutable product of subgroups of A and B, because A does not permute with a subnormal subgroup of order 2 of B.

Theorem. [Ballester-Bolinches, M, Mudziiri Shumba, Pedraza-Aguilera, 2022] Let group $G = AB$ be the weakly mutually *sn*-permutable product of supersoluble subgroups A and B. If A permutes with each Sylow subgroup of B, B permutes with each Sylow subgroup of A and G' is nilpotent, then G is supersoluble.

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Weakly mutually *sn*-permutable products cont

Proof.

- (**Theorem.** If A or B is nilpotent, then G is supersoluble.)
- Proof by contradiction.
- G has a unique minimal normal subgroup N.
- Either $G = AN$ or $G = BN$.
- Either A or B is nilpotent, a contradiction.

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Question. Can we replace supersoluble groups with other classes of finite groups e.g v-supersoluble groups?

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