Simulation of earthquake induced oscillations in a vertical structure using the Finite Element Method

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Background

- The effect of wind and earthquake induced oscillations on high-rise structures is of considerable interest.
- Not just be able to withstand oscillations.
- Non-structural components: affect the functionality, large economic consequences, safety and egress concerns.

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- The effect of wind and earthquake induced oscillations on high-rise structures is of considerable interest.
- Not just be able to withstand oscillations.
- Non-structural components: affect the functionality, large economic consequences, safety and egress concerns.

- *Cheng* [CH15]: “many aspects of building motions can be understood in the context of a simple cantilevered beam”.
- *Picardo* [PTL19]: “The reduction of complex structural systems to equivalent beam models remains an open challenge, of great interest especially in the dynamic field as for the response statistics to wind loads”.

Background - beam models

- Simplified beam models have been used to simulate buildings in a number of recent publications: Twin-beam model (e.g. [MT05] and [HH19]), shear model, flexural model, shear-flexural model and Timoshenko model (e.g. [CH15], [TGA17] and [PTL19]).
- [CH15]: Timoshenko beam model which takes soil-structure interaction into account.
- [TGA17]: concern for using simplified models, suggest using either a coupled beam or Timoshenko.
- An extensive analysis done in *Picardo* [PTL19] to determine under which conditions a simplified model such as the shear beam, bending beam or Timoshenko beam will render accurate results.
- Reliable data!

Outline

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Focus: earthquake induced oscillations of high-rise buildings, modelled by an adapted Timoshenko beam model. (Also relevant for wind induced oscillations - equivalent problem with distributed load.)

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Foundation:

- Rigid body attached to the bottom of the building, moments and shear forces transmitted between the building and foundation.
- Depth below the surface taken into account, NOT regarded as a single point.

Introduction

Motion:

- Ground level motion can be modelled in more than one way. For instance, it can be modelled as a moving foundation or it can be modelled as a force acting on the foundation.
- Our model: Take the earthquake **force** exerted on the building into account. *Huergo* [HH20]: moving soil creates a force which acts on the foundation.
- NB: Take into account that the soil in which the foundation is laid is not rigid.

Beam model for a building

Dimensionless:

- Well known advantages - e.g. less parameters, scaled.
- Additionally: building is not a homogeneous beam, but modelled that way. Floors, supports, different types of material (metal, wood etc.), and the cross-sectional areas might also differ.
- Therefore, attaching meaning to the normal parameters for beams ρ, A, I, E and G is not helpful. Rather combinations of them.

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Assumption:

Vertical (or axial) vibration in the building can be neglected. In the literature considered, only transverse vibrations were taken into account.

Huergo [HH20]: "literature usually neglects the axial vibration when only the lateral response of buildings is studied". ([CH15] and [TGA17]).

Soil-structure interaction

- We use the model for soil structure interaction (SSI) as in *Cheng* [CH15] and *Taciroglu*. [TGA17].
- [TGA17]: "Soil-structure interaction can significantly affect the natural frequencies of a building."
- SSI is not "passive" - explains the mechanism of the earth moving the building (dynamic forcing).
- The mass of the foundation is also taken into account.
- Two springs are incorporated: The elastic effect of the soil-structure interaction is modelled by a "**translational spring** with stiffness K_T" at the base of the building; The angle of rotation is modelled by a "**rotational spring** with stiffness K_R" at the base of the building.

Variables:

- w displacement (transverse)
- ϕ rotation of cross-section
- V shear force
- M moment
- N axial force (due to gravity)

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Boundary conditions: At the **top** of the building ($x = 1$)

$$V(1,t) = M(1,t) = N(1) = 0.$$

At the **base** ($x = 0$)

$$\begin{aligned} M(0,t) &= \gamma^* \phi(0,t) \\ 0 &= -kw(0,t) + V(0,t), \end{aligned}$$

where $\gamma^* = \dfrac{K_R}{\ell AG\kappa^2}$ and $k = \dfrac{K_T \ell}{AG\kappa^2}$ are dimensionless parameters.

Model problem

$$\partial_t^2 w = \partial_x V + Q + \partial_x (N \partial_x w). \quad (1)$$

$$\frac{1}{\alpha}\, \partial_t^2 \phi = V + \partial_x M, \quad (2)$$

$$M = \frac{1}{\beta}\, \partial_x \phi, \quad (3)$$

$$V = \partial_x w - \phi \quad (4)$$

$$N(x) = -\mu(1 - x) \quad (5)$$

Boundary conditions

Top ($x = 1$): $V(1, t) = M(1, t) = N(1) = 0$.

Base: $M(0, t) = \gamma^* \phi(0, t)$ and $0 = -kw(0, t) + V(0, t)$.

Interface condition

$m\partial_t^2 w(0, t) = -k(w(0, t) - E(t)) + V(0, t) + N(0)\partial_x w(0, t).$

Variational form

$$\int_0^1 \partial_t^2 w(\cdot,t)\, z = \int_0^1 \partial_x V(\cdot,t)\, z + \int_0^1 \partial_x(N\partial_x w(\cdot,t))\, z$$
$$= -\int_0^1 V(\cdot,t)\, z' - \int_0^1 N\partial_x w(\cdot,t)\, z'$$
$$- \left(V(0,t) + N(0)\partial_x w(0,t)\right) z(0) \qquad (6)$$

and

$$\int_0^1 \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)\, \psi = \int_0^1 V(\cdot,t)\, \psi + \int_0^1 \partial_x M(\cdot,t)\, \psi$$
$$= \int_0^1 V(\cdot,t)\, \psi - \int_0^1 M(\cdot,t)\, \psi' - M(0,t)\psi(0), \qquad (7)$$

with

$$m\partial_t^2 w(0,t) = -k(w(0,t) - E(t)) + V(0,t) + N(0)\partial_x w(0,t). \qquad (8)$$

Simulation of motion

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Galerkin approximation

Divide the interval $[0, 1]$ into n elements of equal length.
In the numerical experiments Hermite piecewise cubic basis functions $\delta_1, \delta_2, \ldots, \delta_{2n+2}$ are used.
These basis functions span the finite dimensional subspace S^h.

Galerkin approximation

Consider approximations w_h and ϕ_h.

$$w_h = \sum_{j=1}^{2n+2} w_j(t)\delta_j(x) \text{ and } \phi_h = \sum_{j=1}^{2n+2} \phi_j(t)\delta_j(x).$$

Notation: $(u, v) = \int_0^1 uv\,dx$ and dots used for time derivatives.

Find w_h, ϕ_h in S^h such that

$$\sum_{j=1}^{2n+2} \ddot{w}_j(\delta_j, \delta_i) = -\sum_{j=1}^{2n+2} \left(w_j(\delta_j', \delta_i') - \phi_j(\delta_j, \delta_i')\right) - \sum_{j=1}^{2n+2} w_j \left(-\mu(1 - x_i)\delta_j', \delta_i'\right)$$
$$- k \sum_{j=1}^{2n+2} w_j\delta_j(0)\delta_i(0) + kE(t)\delta_i(0) - m \sum_{j=1}^{2n+2} \ddot{w}_j\delta_j(0)\delta_i(0),$$

for $i = 1, \ldots, 2n+2$. Note $\delta_i(0) = 0$ unless $i = 1$.

Matrix notation

Define the following FEM matrices: for $i, j = 1, \ldots, 2n+2$,

$$K_{ij} = (\delta_j', \delta_i'),\ \ M_{ij} = (\delta_j, \delta_i),\ \ L_{ij} = (\delta_j, \delta_i').$$

First equation of motion in matrix notation:

$$M\ddot{\bar{w}} = -\left(K\bar{w} - L\bar{\phi}\right) - \tilde{N}\bar{w} - kR\bar{w} + kE(t)R^1_{col} - mR\ddot{\bar{w}},$$

$$\text{where} \quad R_{ij} = 1 \text{ for } i = j = 1 \text{ and } 0 \text{ otherwise.}$$

Similarly, for the second equation of motion

$$\frac{1}{\alpha}M\ddot{\bar{\phi}} = L^T\bar{w} - M\bar{\phi} - \frac{1}{\beta}K\bar{\phi} - \gamma^* R\bar{\phi}. \tag{9}$$

Let $\bar{u} = \begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix}$.

$$\begin{bmatrix} M + mR & 0 \\ 0 & \frac{1}{\alpha}M \end{bmatrix} \ddot{\bar{u}} = \begin{bmatrix} -K - \tilde{N} - kR & L \\ L^T & -M - \frac{1}{\beta}K - \gamma^* R \end{bmatrix} \bar{u} + kE(t) \begin{bmatrix} R^1_{col} \\ 0 \end{bmatrix}$$

- This is of the form $A\ddot{\bar{u}} = B\bar{u} + \bar{F}(t)$, finite difference scheme can be applied. We used central difference, average acceleration.
- Programmed in Matlab, convergence experiments done.
- A way to check if correct - compare to ”exact” solution obtained using partial sums.

Eigenvalue problem

Consider pair $w(x,t) = T(t)u(x)$ and $\phi(x,t) = T(t)\psi(x)$ as a possible solution. Leads to the eigenvalue problem

$$\begin{aligned}(1+N)u'' - \psi' + N'u' &= -\lambda u,\\ \alpha u' - \alpha\psi + \frac{\alpha}{\beta}\psi'' &= -\lambda\psi,\end{aligned}$$

with boundary conditions

$$u'(1) - \psi(1) = \psi'(1) = 0,$$
$$\frac{1}{\beta}\psi'(0) = \gamma^*\psi(0),$$
$$-m\lambda u(0) = -ku(0) + u'(0) - \psi(0) + N(0)u'(0).$$

Can be solved analytically or with finite element method.

Find u and ψ such that

$$\int_0^1 u'v' + ku(0)v(0) - \int_0^1 \psi v' = \lambda \left(\int_0^1 uv + mu(0)v(0) \right),$$

$$\int_0^1 \frac{1}{\beta}\psi' z' + \int_0^1 \psi z + \gamma^* \psi(0)z(0) - \int_0^1 u'z = \lambda \int_0^1 \frac{1}{\alpha}\psi z$$

for all v and z in $C^1_+[0,1]$.
A solution of the variational eigenvalue problem above is a pair of functions $\langle u, \psi \rangle \neq 0$ with a corresponding real number λ.
The function T satisfies $T'' = -\lambda T$.

Eigenvalue problem in matrix form

Galerkin approximation in matrix form:

$$\begin{aligned} K\bar{u} + kR\bar{u} - L\bar{\psi} &= \lambda(M\bar{u} + mR\bar{u}), \\ \frac{1}{\beta}K\bar{\psi} + M\bar{\psi} + \gamma^* R\bar{\psi} - L^T\bar{u} &= \lambda\left(\frac{1}{\alpha}M\bar{\psi}\right). \end{aligned}$$

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Now consider the following notation:

$$\bar{e} = \begin{bmatrix} \bar{u} \\ \bar{\psi} \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K + kR & -L \\ -L^T & \frac{1}{\beta}K + M + \gamma^* R \end{bmatrix} \text{ and } \mathcal{M} = \begin{bmatrix} M + mR & 0 \\ 0 & \frac{1}{\alpha}M \end{bmatrix}.$$

The eigenvalue problem is to find a eigenvalue λ and associated eigenvector $\bar{e}$ such that

$$\mathcal{K}\,\bar{e} = \lambda\mathcal{M}\,\bar{e}.$$

We find a sequence of eigenvalues $\{\lambda_j\}$ and eigenvectors $\bar{e}_j$. The eigenvector functions are normalised and participation factors calculated.

Numerical results

- In the experiments we chose $E(t) = P\sin(p\,t)$, where P and p are constants and are the amplitude and frequency of the ground motion respectively.

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- Compare solution obtained using the partial sums of the modes of vibration with the finite element approximation. The two methods produced the same results.
- Investigate transient response.

Natural frequencies

$\alpha = 4800, \gamma = 0.25, \gamma^* = 0.95$, k=0.01	m= 0.01	m=0.05	m= 0.1
$f_1 = \sqrt{\lambda_1}$	0.0751	0.0747	0.0742
$f_2 = \sqrt{\lambda_2}$	0.2080	0.2003	0.1919
$f_3 = \sqrt{\lambda_2}$	0.8620	0.8315	0.8023
$f_4 = \sqrt{\lambda_4}$	2.0612	1.9973	1.9425
$f_5 = \sqrt{\lambda_5}$	3.7184	3.6125	3.5299

Table: Natural frequencies

Comparison of FEM and partial sums

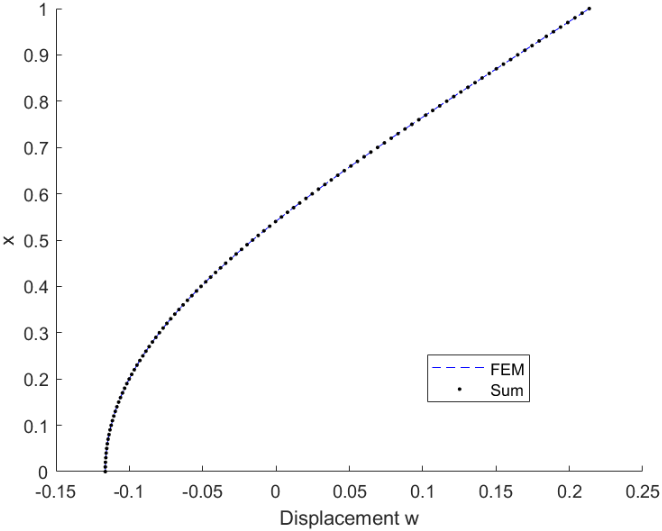
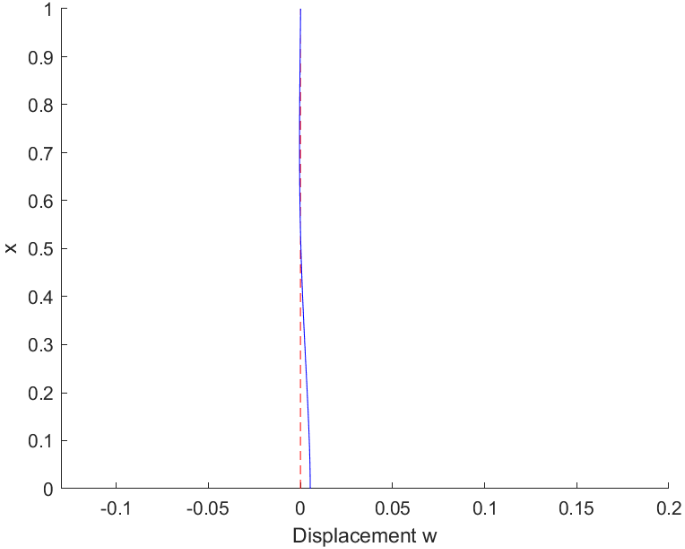
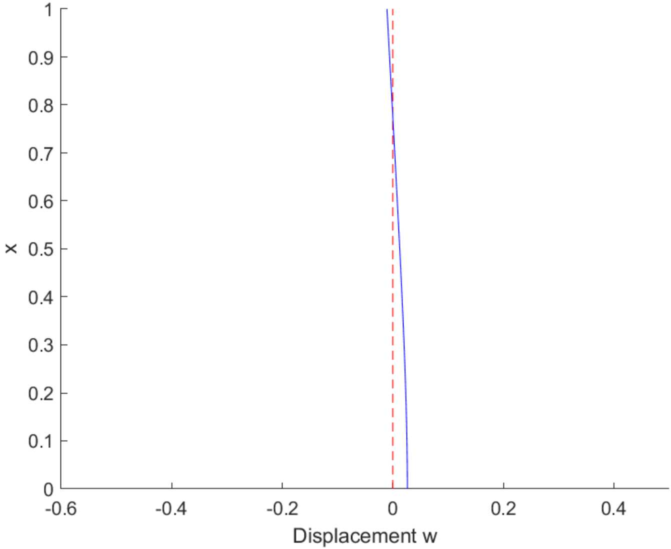


Figure: Comparison of two methods

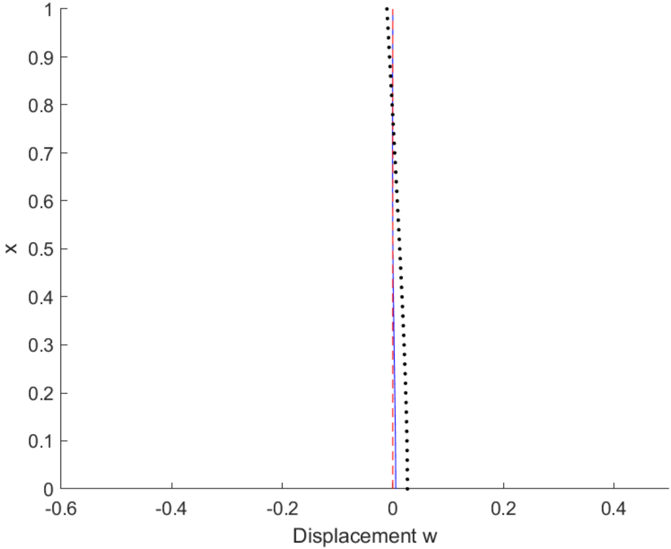
Transient response



Transient response: close to natural freq



Transient response: comparison



Some notes

- Transient response of the structure is illustrated for a full period of the ground motion (denoted by τ_g).
- Note “whiplash” effect at the top of the structure at certain times.
- NB: Scaling!
- Other experiments.
- Future work...

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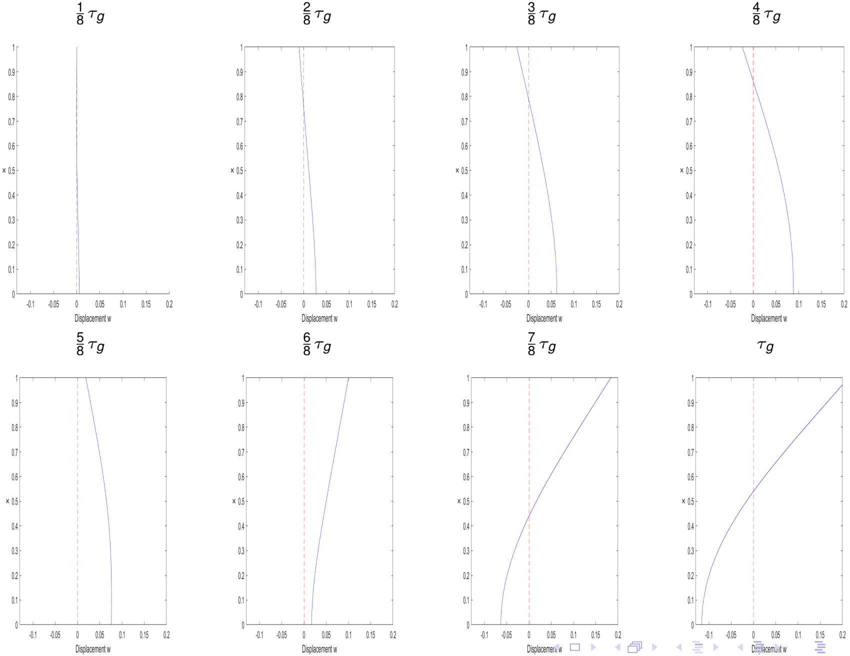
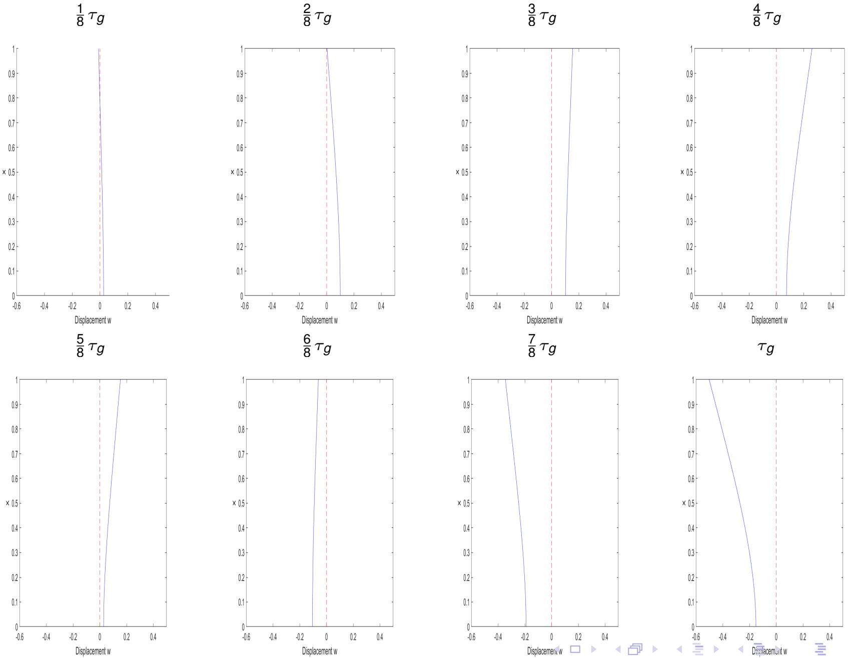
Transient response: p=0.2



Transient response: p=0.07 (close to natural freq)



Transient response: comparison