The Mincut Graph of a Graph

C.W. Kriel E.G. Mphako-Banda

School of Mathematics
University of the Witwatersrand

SAMS December 2022

Outline

1. Introduction
   - Intersection graphs and graph operators
2. The mincut graph and mincut operator
   - Main Results
   - Basic Outlines for some Proofs
   - Conjectures and further questions

Definition

A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Definition

A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Definition

Given a set S and a family $F = \{S_1, S_2 \ldots, S_i\}$ of subsets of S, an *intersection graph* of F is a graph with vertices v_i corresponding to each of the S_i and two vertices v_i and v_j are adjacent if $S_i \cap S_j \neq \emptyset$

Definition

A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Definition

Given a set S and a family $F = \{S_1, S_2 \ldots, S_i\}$ of subsets of S, an *intersection graph* of F is a graph with vertices v_i corresponding to each of the S_i and two vertices v_i and v_j are adjacent if $S_i \cap S_j \neq \emptyset$

- Every graph is an intersection graph (Szpilrajn-Marczewski, 1945)

Definition

A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Definition

Given a set S and a family $F = \{S_1, S_2 \ldots, S_i\}$ of subsets of S, an *intersection graph* of F is a graph with vertices v_i corresponding to each of the S_i and two vertices v_i and v_j are adjacent if $S_i \cap S_j \neq \emptyset$

- Every graph is an intersection graph (Szpilrajn-Marczewski, 1945)
- One of the first class of intersection graphs to be widely studied was the line graph.

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- If ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$.

Definition

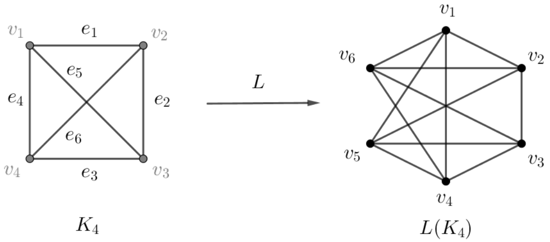
A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- If ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$.
- The Line Graph: Whitney (1935) and Krausz (1943). The line graph operator L maps every graph G to its line graph $L(G)$.

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- If ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$.
- The Line Graph: Whitney (1935) and Krausz (1943). The line graph operator L maps every graph G to its line graph $L(G)$.



The Mincut Graph

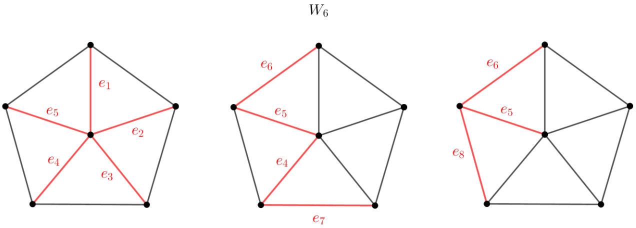
Definition

Let G be a simple connected graph, then an edge-cut of G is a subset X of $E(G)$, such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a *minimum edge-cut* and this cardinality is the edge-connectivity of G, denoted $\lambda(G)$. We will call such a minimum edge-cut a *mincut* of G.

The Mincut Graph

Definition

Let G be a simple connected graph, then an edge-cut of G is a subset X of $E(G)$, such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a *minimum edge-cut* and this cardinality is the edge-connectivity of G, denoted $\lambda(G)$. We will call such a minimum edge-cut a *mincut* of G.



The Mincut Graph

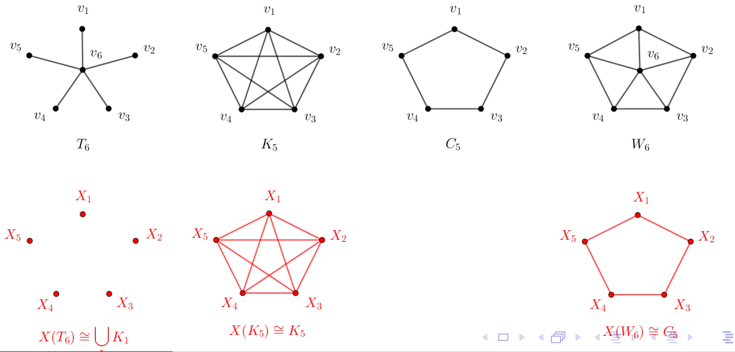
Definition

Let $X = \{X_1, X_2, \dots X_i\}$ be the set of all mincuts of a simple connected graph G. Represent each of the X_i with a vertex v_i such that two vertices v_i and v_j are adjacent if $X_i \cap X_j \neq \emptyset$, and call this graph the *mincut graph* of G, denoted by $X(G)$.

The Mincut Graph

Definition

Let $X = \{X_1, X_2, \ldots X_i\}$ be the set of all mincuts of a simple connected graph G. Represent each of the X_i with a vertex v_i such that two vertices v_i and v_j are adjacent if $X_i \cap X_j \neq \emptyset$, and call this graph the *mincut graph* of G, denoted by $X(G)$.

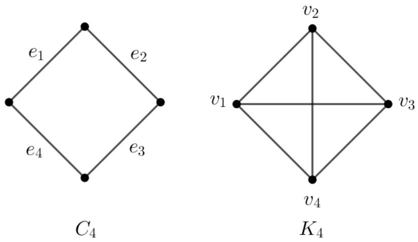


Mincut graphs of some graph classes

- $X(C_4) \cong L(K_4)$

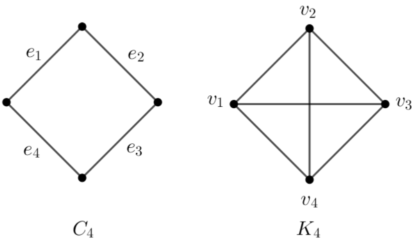
Mincut graphs of some graph classes

- $X(C_4) \cong L(K_4)$



Mincut graphs of some graph classes

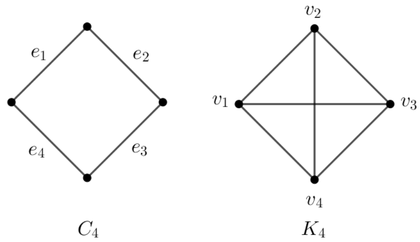
- $X(C_4) \cong L(K_4)$



- For C_4,
  $X = \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\}$

Mincut graphs of some graph classes

- $X(C_4) \cong L(K_4)$



- For C_4,
  $X = \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\}$
- For K_4,
  $E(K_4) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator
- Research programme on graph operators, as introduced by Prisner in the 1995 monograph "Graph Dynamics"

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator
- Research programme on graph operators, as introduced by Prisner in the 1995 monograph "Graph Dynamics"
- Questions:

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator
- Research programme on graph operators, as introduced by Prisner in the 1995 monograph “Graph Dynamics”
- Questions:
  - Which graphs appear as images of graphs? (When is a graph a mincut graph?)

The Mincut Graph Operator

Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator
- Research programme on graph operators, as introduced by Prisner in the 1995 monograph “Graph Dynamics”
- Questions:
  - Which graphs appear as images of graphs? (When is a graph a mincut graph?)
  - Which graphs are fixed under the operator?

The Mincut Graph Operator

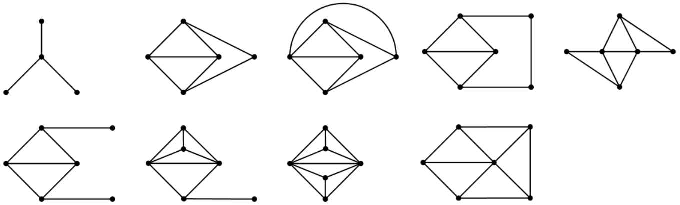
Definition

A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

- The mincut graph as a graph operator
- Research programme on graph operators, as introduced by Prisner in the 1995 monograph "Graph Dynamics"
- Questions:
  - Which graphs appear as images of graphs? (When is a graph a mincut graph?)
  - Which graphs are fixed under the operator?
  - What happens when the operator is iterated?

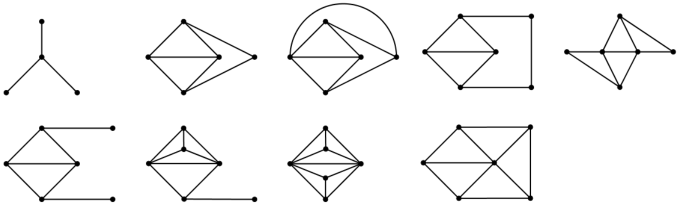
The Line graph operator: Wilf and van Rooij (1965) and Beineke (1967)

- A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, including $K_{1,3}$.



The Line graph operator: Wilf and van Rooij (1965) and Beineke (1967)

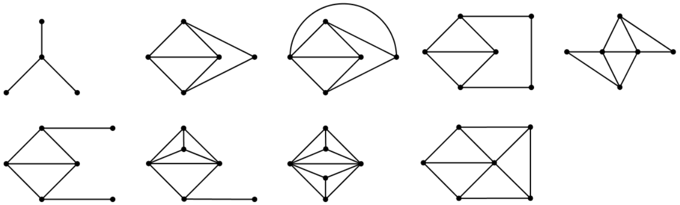
- A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, including $K_{1,3}$.



- $C_n \cong L(C_n)$

The Line graph operator: Wilf and van Rooij (1965) and Beineke (1967)

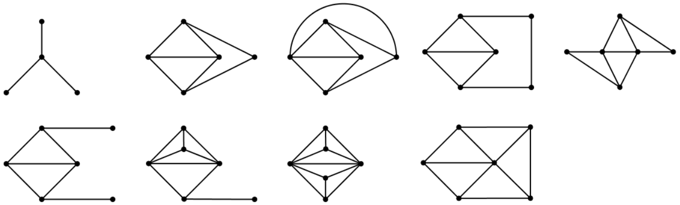
- A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, including $K_{1,3}$.



- $C_n \cong L(C_n)$
- Repeated application of L leads to steadily increasing numbers of vertices unless a finite number of conditions apply:

The Line graph operator: Wilf and van Rooij (1965) and Beineke (1967)

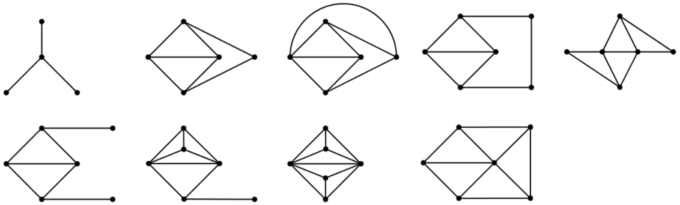
- A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, including $K_{1,3}$.



- $C_n \cong L(C_n)$
- Repeated application of L leads to steadily increasing numbers of vertices unless a finite number of conditions apply:
  - G is a cycle or $K_{1,3}$

The Line graph operator: Wilf and van Rooij (1965) and Beineke (1967)

- A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, including $K_{1,3}$.



- $C_n \cong L(C_n)$
- Repeated application of L leads to steadily increasing numbers of vertices unless a finite number of conditions apply:
  - G is a cycle or $K_{1,3}$
  - G is a path.

Main Results

- Every graph is a mincut graph

Main Results

- Every graph is a mincut graph
- Graphs fixed under the operator are r-regular and super-λ

Main Results

- Every graph is a mincut graph
- Graphs fixed under the operator are r-regular and super-λ
- 2-periodic graphs, $X^2(G) \cong G$

Main Results

- Every graph is a mincut graph
- Graphs fixed under the operator are r-regular and super-λ
- 2-periodic graphs, $X^2(G) \cong G$
- No graph diverges under iteration of the operator.

Every graph is a mincut graph

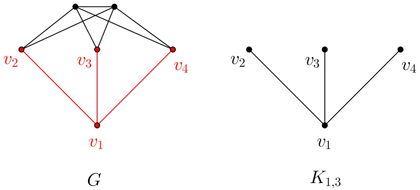
Lemma

Let G be a super-λ graph with $H \subseteq G$ the induced subgraph on the set of vertices of G such that $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$, then $H \cong X(G)$.

Every graph is a mincut graph

Lemma

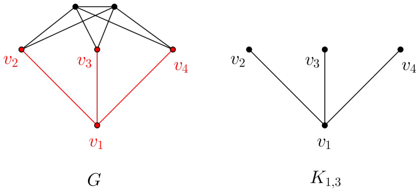
Let G be a super-λ graph with $H \subseteq G$ the induced subgraph on the set of vertices of G such that $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$, then $H \cong X(G)$.



Every graph is a mincut graph

Lemma

Let G be a super-λ graph with $H \subseteq G$ the induced subgraph on the set of vertices of G such that $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$, then $H \cong X(G)$.



Corollary

If G is super-λ and r-regular, then $X(G) \cong G$.

Every graph is a mincut graph

Theorem

Every graph G is the mincut graph of a family of graphs.

Every graph is a mincut graph

Theorem

Every graph G is the mincut graph of a family of graphs.

Proof.

We construct a super graph G' that is super-λ with $G \subseteq G'$ the induced subgraph on the vertices of G' such that $V(G) = \{v \in V(G') | deg(v) = \delta(G')\}$. □

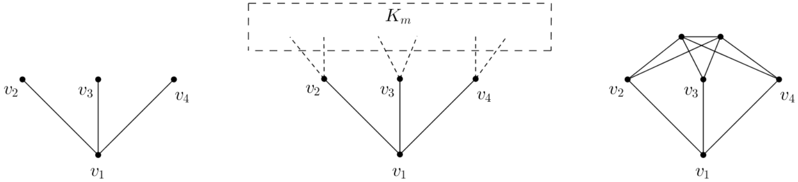
Every graph is a mincut graph

Theorem

Every graph G is the mincut graph of a family of graphs.

Proof.

We construct a super graph G' that is super-λ with $G \subseteq G'$ the induced subgraph on the vertices of G' such that $V(G) = \{v \in V(G') | deg(v) = \delta(G')\}$. □



$K_{1,3}$ G' G'_1

Roots and Depth

Definition

For a given graph operator ϕ, a ϕ-*root* of a graph G is any graph H with $\phi(H) \cong G$.

Roots and Depth

Definition

For a given graph operator ϕ, a ϕ-*root* of a graph G is any graph H with $\phi(H) \cong G$.

Corollary

Every graph G has an infinite number of X-roots.

Roots and Depth

Definition

For a given graph operator ϕ, a ϕ-*root* of a graph G is any graph H with $\phi(H) \cong G$.

Corollary

Every graph G has an infinite number of X-roots.

Definition

For a graph G and an operator ϕ we define the ϕ-depth, $depth(G)$, as the largest integer (if there is one, otherwise ∞) for which there is some graph H such that $\phi^d(H) \cong G$.

Roots and Depth

Definition

For a given graph operator ϕ, a ϕ-*root* of a graph G is any graph H with $\phi(H) \cong G$.

Corollary

Every graph G has an infinite number of X-roots.

Definition

For a graph G and an operator ϕ we define the ϕ-depth, $depth(G)$, as the largest integer (if there is one, otherwise ∞) for which there is some graph H such that $\phi^d(H) \cong G$.

Corollary

Every graph G has infinite X-depth.

Graphs fixed under the operator

Theorem

$G \cong X(G)$ *if, and only if,* G *is super-*λ *and* r*-regular.*

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

□

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

- Sufficiency already proven: G is super-λ and r-regular implies $G \cong X(G)$.

□

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

- Sufficiency already proven: G is super-λ and r-regular implies $G \cong X(G)$.
- Necessity:

□

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

- Sufficiency already proven: G is super-λ and r-regular implies $G \cong X(G)$.
- Necessity:
  - Every mincut X partitions $V(G)$ into two vertex sets $\langle A, \overline{A} \rangle$.

□

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

- Sufficiency already proven: G is super-λ and r-regular implies $G \cong X(G)$.
- Necessity:
  - Every mincut X partitions $V(G)$ into two vertex sets $\langle A, \overline{A} \rangle$.
  - If $G \cong X(G)$, then for every $v \in V(G)$ there is a mincut $X \subset E(G)$ such that, if $v_i v_j \in E(G)$ then $X_i \cap X_j \neq 0$.

□

Graphs fixed under the operator

Theorem

$G \cong X(G)$ if, and only if, G is super-λ and r-regular.

Proof.

- Sufficiency already proven: G is super-λ and r-regular implies $G \cong X(G)$.
- Necessity:
  - Every mincut X partitions $V(G)$ into two vertex sets $\langle A, \overline{A} \rangle$.
  - If $G \cong X(G)$, then for every $v \in V(G)$ there is a mincut $X \subset E(G)$ such that, if $v_i v_j \in E(G)$ then $X_i \cap X_j \neq 0$.
  - If G is not super-λ then there is at least one non-trivial mincut $X \subset E(G)$ and, hence, if $G \cong X(G)$ then there is at least one $v \in V(G)$ such that $deg(v) > \lambda$.

□

Graphs fixed under the operator

Definition

Let $X = \langle A, \overline{A} \rangle$ and $Y = \langle B, \overline{B} \rangle$ be two mincuts of a graph G. If their vertex sets have non-empty intersection, that is $A \cap B \neq \emptyset$, then they are either *nested*, i.e. $A \subset B$ or $B \subset A$, or they *overlap* (also called *crossing mincuts*), i.e. $A \cap B$, $\overline{A} \cap B$ and $A \cap \overline{B}$ are *non-empty*.

Graphs fixed under the operator

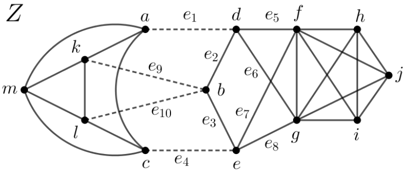
Definition

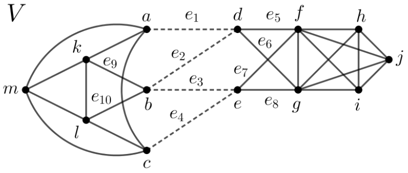
Let $X = \langle A, \overline{A} \rangle$ and $Y = \langle B, \overline{B} \rangle$ be two mincuts of a graph G. If their vertex sets have non-empty intersection, that is $A \cap B \neq \emptyset$, then they are either *nested*, i.e. $A \subset B$ or $B \subset A$, or they *overlap* (also called *crossing mincuts*), i.e. $A \cap B$, $\overline{A} \cap B$ and $A \cap \overline{B}$ are *non-empty*.

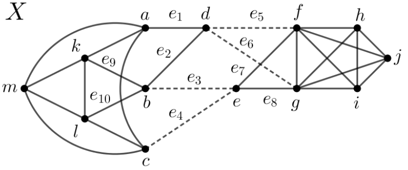
Lemma (Chandran & Ram)

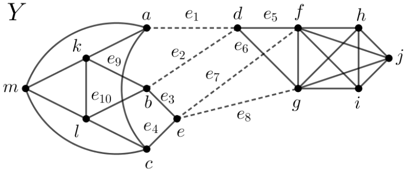
If $X = \langle A, \overline{A} \rangle$ and $Y = \langle B, \overline{B} \rangle$ are a pair of crossing mincuts, then $X \cap Y = \emptyset$.

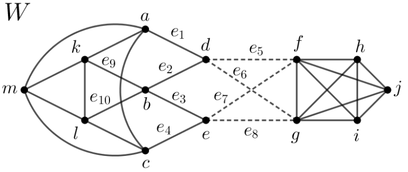
Graphs fixed under the operator











Iteration of the operator

- Periodicity:

Iteration of the operator

- Periodicity:

Definition

A graph G is k-periodic in ϕ if $\phi^k(G) \cong G$ and $\phi^i(G) \ncong G$ for $1 < i < k$.

Iteration of the operator

- Periodicity:

Definition

A graph G is k-periodic in ϕ if $\phi^k(G) \cong G$ and $\phi^i(G) \ncong G$ for $1 < i < k$.

  - Graphs of period 1 (fixed)

Iteration of the operator

- Periodicity:

Definition

A graph G is k-periodic in ϕ if $\phi^k(G) \cong G$ and $\phi^i(G) \ncong G$ for $1 < i < k$.

  - Graphs of period 1 (fixed)
  - Graphs of period 2

Iteration of the operator

- Periodicity:

Definition

A graph G is k-periodic in ϕ if $\phi^k(G) \cong G$ and $\phi^i(G) \ncong G$ for $1 < i < k$.

- Graphs of period 1 (fixed)
- Graphs of period 2

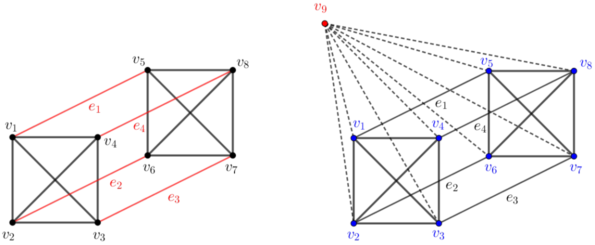


Figure: $K_n \times K_2,\ n > 2$

Iteration of the operator

Definition

Let G be a cycle on n vertices and replace each vertex with K_m such that m is even and $m > 2$. Connect $m/2$ vertices from each complete component to $m/2$ corresponding vertices in each of the two adjacent complete components and delete the original edges of the cycle such that each vertex in the new graph has degree m. We call this new graph an (m, n)-*complete component cycle* and denote it by $C_{m,n}$.

Iteration of the operator

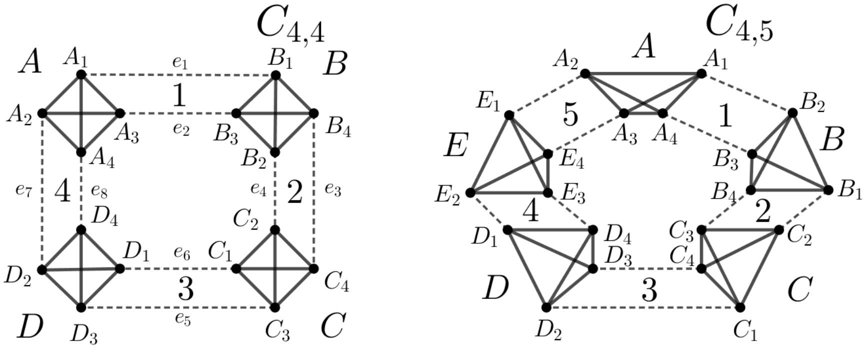


Figure: $C_{4,4}$ and $C_{4,5}$.

Iteration of the operator

- Convergence

Iteration of the operator

- Convergence

Definition

Let G be a graph and ϕ an operator on G. G is *convergent* if $\{\phi^n(G) | n \in \mathbb{N}\}$ is finite. That is, G is ϕ-convergent if and only if some iterated ϕ-graph is ϕ-periodic. If G is not convergent then it is *divergent*.

Iteration of the operator

- Convergence

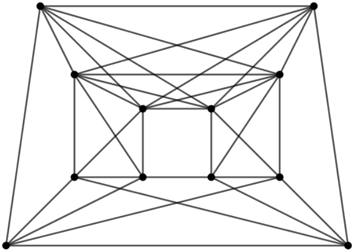
Definition

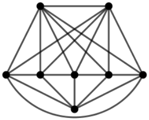
Let G be a graph and ϕ an operator on G. G is *convergent* if $\{\phi^n(G) | n \in \mathbb{N}\}$ is finite. That is, G is ϕ-convergent if and only if some iterated ϕ-graph is ϕ-periodic. If G is not convergent then it is *divergent*.

Theorem

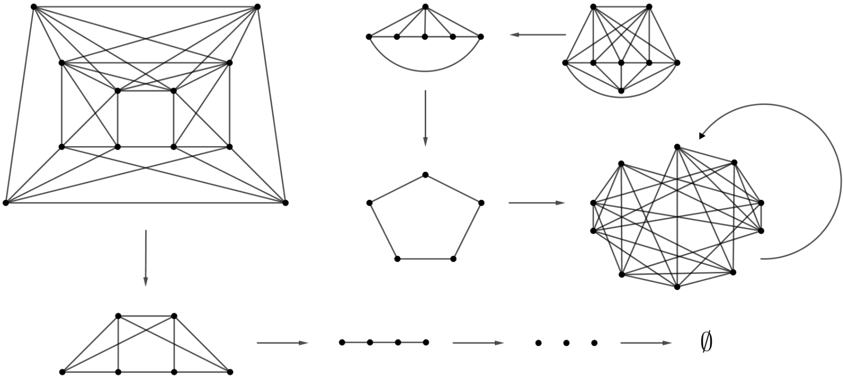
Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Iteration of the operator





Iteration of the operator



Iteration of the operator

Theorem

Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Iteration of the operator

Theorem

Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Proof.

□

Iteration of the operator

Theorem

Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Proof.

- Suppose n increases but m does not. If the graph becomes sufficiently *sparse* then $X^i(G)$ converges to the null graph.

Iteration of the operator

Theorem

Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Proof.

- Suppose n increases but m does not. If the graph becomes sufficiently *sparse* then $X^i(G)$ converges to the null graph.
- Suppose m increases but n does not. If the graph becomes sufficiently *dense* $X^i(G)$ becomes fixed.

□

Iteration of the operator

Theorem

Let G be a simple graph of order n and size m with minimum degree δ and minimum edge-connectivity λ then G converges under iteration of the X-operator, that is $X^i(G)$ converges for sufficiently large i.

Proof.

- Suppose n increases but m does not. If the graph becomes sufficiently *sparse* then $X^i(G)$ converges to the null graph.
- Suppose m increases but n does not. If the graph becomes sufficiently *dense* $X^i(G)$ becomes fixed.
- Hence, we need both n and m to increase under iteration of the operator in order for the graph to diverge. If $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$, or equivalently, $deg(u) + deg(v) \geq n$ for any $u, v \in V(G)$, $uv \notin E(G)$ then G is super-λ or $G \cong K_{n/2} \times K_2$.

□

Conjectures and further questions

Conjecture (Convergence to null graph)

Let G be a simple connected graph and $X(\cdot)$ the mincut operator. Then $X^k(G) \to \emptyset$ except in a finite number of cases.

Further questions

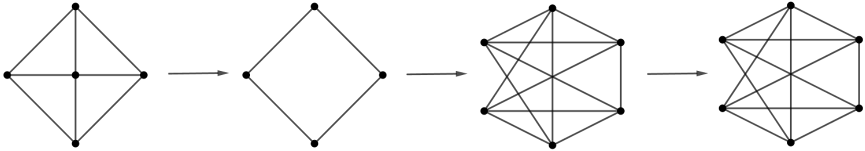
- Periodicity

Further questions

- Periodicity
- Connected mincut graphs

Further questions

- Periodicity
- Connected mincut graphs
- Reconstruction problem



Thank you.