Attractors of Iterated Function System and its Applications

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Definition: Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be **metric** if it satisfies the following:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

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Example:

- $X = \mathbb{R}$ with $d(x, y) = |x - y|$ is metric space.
- $X = \mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. d is known as Euclidean metric.

Definition:

- Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be Cauchy if and only if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$.

Preliminaries

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- Let (X, d) be a metric space. A mapping $f : X \to X$ is called **contraction** on X if there exists a positive real number $\alpha < 1$ such that

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- A **fixed point** of a self mapping $f : X \to X$ is an $x \in X$ which is mapped onto itself, that is,

$$fx = x.$$

In other words $x \in X$ remains fixed under f.

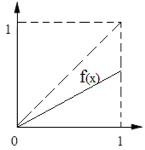
Banach Contraction Theorem

Theorem Let (X, d) be a complete metric space and let the mapping $f : X \to X$ be contraction on X with contraction constant α, that is, if there exists a positive real number $\alpha < 1$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \text{for all} \quad x, y \in X.$$

Then f has a unique fixed point. Furthermore, for every $x \in X$ the sequence of iterates $\{x, fx, f^2x, f^3x, \cdots\}$ converges to fixed point of f.

Example: Let $X = [0,1]$ with usual metric and let $f : X \to X$ be defined by $fx = \frac{x}{2}$ for all $x \in X$.



f is a contraction with $\alpha = \frac{1}{2}$

$$d(fx, fy) = |\frac{x}{2} - \frac{y}{2}| = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y)$$

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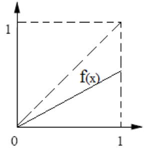

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Contraction mappings

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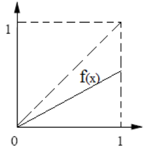

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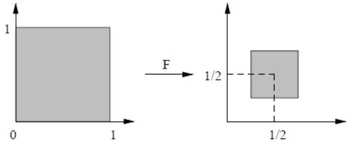
- $x = \frac{1}{3}$, $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots\} \to 0$.
- $x = \frac{3}{4}$, $\{\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \cdots\} \to 0$.

Example: Let $X = [0,1]^2$ with Euclidean metric d and let $f : X \to X$ be defined by $f(x,y) = (\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{4})$;$(x,y) \in [0,1]^2$.



Let $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in [0,1]^2$, then

$$d(f(P_1), f(P_2)) = \sqrt{(\frac{x_1}{2} + \frac{1}{4} - \frac{x_2}{2} - \frac{1}{4})^2 + (\frac{y_1}{2} + \frac{1}{4} - \frac{y_2}{2} - \frac{1}{4})^2}$$
$$= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \frac{1}{2} d(P_1, P_2)$$

Hence, f is a contraction with $\alpha = \frac{1}{2}$ and has a unique fixed point $(\frac{1}{2}, \frac{1}{2})$.

Hausdorff distance

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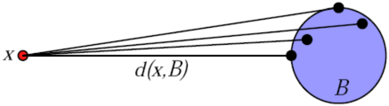
- In $\mathbb{R}^n$ closed and bounded subsets are compact.
  - Every closed interval and every finite set in $\mathbb{R}$ are compact.

Let $\mathcal{C}(X)$ denotes the set of all non-empty compact subsets of a complete metric space (X, d). For $A, B, C \in \mathcal{C}(X)$.

Definition: If $x \in X$, the *distance* form x to B is

$$d(x, B) = \min\{d(x, y) : y \in B\} \tag{1}$$



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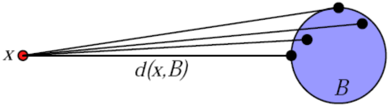

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$d(x, B) = \min\{d(3, y) : y \in [0, 1]\} = |3 - 1| = 2.$

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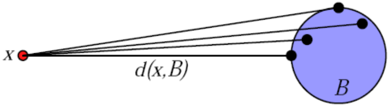

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1. $x \in B \implies d(x, B) = 0.$
2. $B \subset C \implies d(x, C) \leq d(x, B).$

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$$\begin{aligned} d(A, B) &= \max\{d(x, [3, 5]) : x \in [0, 1]\} \\ &= \max\{|x - 3| : x \in [0, 1]\} = |0 - 3| = 3. \end{aligned}$$

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Similarly,

$$\begin{aligned} d(B, A) &= \max\{d(y, [0, 1]) : y \in [3, 5]\} = d(3, [0, 1]) \\ &= \min\{d(3, x) : x \in [0, 1]\} = |3 - 1| = 2. \end{aligned}$$

Hence, $d(A, B) \neq d(B, A)$.

Hausdorff distance

The *distance* from A to B is

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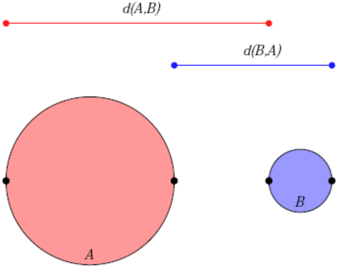
Note that d is not a metric on $\mathcal{C}(X)$, since it is not symmetric in general that is $d(A, B) \neq d(B, A)$.



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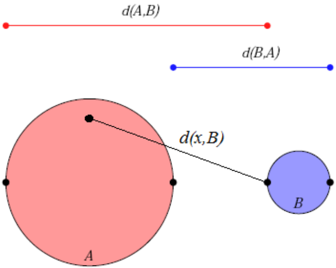

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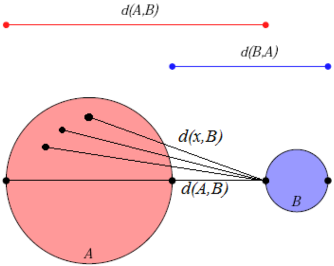

Lemma 1.1: Let $A, B, C \in \mathcal{C}(X)$ then,

1. $B \subset C \implies d(A, C) \leq d(A, B)$,
2. $d(A \cup B, C) = \max\{d(A, C), d(B, C)\}$.

Proof: Since, $B \subset C$ then

$$d(x, C) \leq d(x, B) \quad \text{for all } x \in A.$$

$$\implies d(A, C) \leq d(A, B).$$

(2) follows from

$$\begin{aligned} d(A \cup B, C) &= \max\{d(x, C) : x \in A \cup B\} \\ &= \max\{\max_{x \in A} d(x, C), \max_{x \in B} d(x, C)\} \\ &= \max\{d(A, C), d(B, C)\}. \end{aligned}$$

Definition: Let (X, d) be a complete metric space, the **Hausdorff** distance between A and B in $\mathcal{C}(X)$ is defined by

$$h(A, B) = \max\{d(A, B), d(B, A)\}. \tag{2}$$

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Lemma 1.2: For all $A, B, C, D \in \mathcal{C}(X)$,

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\} \tag{3}$$

Proof: By using (2) of Lemma 1.1, we have

$$\begin{aligned} d(A \cup B, C \cup D) &= \max\{d(A, C \cup D), d(B, C \cup D)\} \\ &\leq \max\{d(A, C), d(B, D)\} \text{ (using (1) of Lemma 1.1)} \\ &\leq \max\{h(A, C), h(B, D)\}. \end{aligned}$$

The same argument yields,

$$d(C \cup D, A \cup B) \leq \max\{h(A, C), h(B, D)\}. \tag{4}$$

Theorem: Let (X, d) be a complete metric space. Then the mapping $h : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ defines a metric on $\mathcal{C}(X)$. Furthermore, $(\mathcal{C}(X), h(d))$ is complete.

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Let $A, B, C \in \mathcal{C}(X)$

1. $h(A, A) = d(A, A) = \max\{d(x, A) : x \in A\} = 0$,
   If $A \neq B \implies \exists a \in A$ such that $a \notin B$.
   It follows that $h(A, B) \geq d(A, B) > 0$.
2. $h(A, B) = h(B, A)$
3. We first show that $d(A, B) \leq d(A, C) + d(C, B)$.

Hausdorff distance

For any $a \in A$ we have,

$$\begin{aligned} d(a,B) &= \min\{d(a,b) : b \in B\} \\ &\leq \min\{d(a,c) + d(c,b) : b \in B\} \quad \forall c \in C \\ &= d(a,c) + \min\{d(c,b) : b \in B\} \quad \forall c \in C \\ &= d(a,c) + d(c,B) \quad \forall c \in C \end{aligned}$$

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$$\begin{aligned} h(A,B) &= \max\{d(A,B), d(B,A)\} \\ &\leq \max\{d(A,C) + d(C,B), d(B,C) + d(C,A)\} \\ &= \max\{d(A,C), d(C,A)\} + \max\{d(C,B), d(B,C)\} \\ &= h(A,C) + h(C,B). \end{aligned}$$

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- $f : \mathcal{C}(X) \to \mathcal{C}(X)$ defined by

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Proof: It is obvious that every contraction mapping is continuous. Further under every continuous mapping $f : X \to X$, the image of a compact subset is also compact. That is,

$$A \in \mathcal{C}(X) \implies f(A) \in \mathcal{C}(X).$$

Now let $A, B \in \mathcal{C}(X)$.

Contraction on Hausdorff metric

$$\begin{aligned} d(f(A), f(B)) &= \max\{d(fx, f(B)) : x \in A\} \\ &= \max\{\min\{d(fx, fy) : y \in B\} : x \in A\} \\ &\leq \max\{\min\{\alpha.d(x, y) : y \in B\} : x \in A\} \\ &= \alpha.d(A, B). \end{aligned}$$

Similarly, $d(f(B), f(A)) \leq \alpha.d(B, A)$.

$$\begin{aligned} \text{Hence, } h(f(A), f(B)) &= \max\{d(f(A), f(B)), d(f(B), f(A))\} \\ &\leq \max\{\alpha d(A, B), \alpha d(B, A)\} \\ &= \alpha h(A, B). \end{aligned}$$

Hence $f : \mathcal{C}(X) \to \mathcal{C}(X)$ is contraction with same constant α.

Contraction on Hausdorff metric

Theorem: Let (X, d) be a metric space. Let $\{f_n : n = 1, 2, \cdots, N\}$ be contraction mappings on $(\mathcal{C}(X), h)$. Let the contraction constant for f_n be denoted by α_n for each n. Define $F : \mathcal{C}(X) \to \mathcal{C}(X)$ by

$$\begin{aligned} F(A) &= f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) \\ &= \cup_{n=1}^{N} f_n(A), \quad \text{for each } A \in \mathcal{C}(X). \end{aligned}$$

Then F is a contraction mapping with contraction constant $\alpha = \max\{\alpha_n : n = 1, 2, \cdots, N\}$.

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Proof: We demonstrate the claim for $N = 2$. Let $A, B \in \mathcal{C}(X)$, then using Lemma 1.2, we have

$$\begin{aligned} h(F(A), F(B)) &= h(f_1(A) \cup f_2(A), f_1(B) \cup f_2(B)) \text{ (Lemma 1.2)} \\ &\le \max\{h(f_1(A), f_1(B)), h(f_2(A), f_2(B))\} \\ &\le \max\{\alpha_1 h(A, B), \alpha_2 h(A, B)\} \le \alpha h(A, B). \end{aligned}$$

Iterated function system (IFS)

- The **Iterated function system** or **IFS** consists of a complete metric space (X, d) together with a finite set of contraction mappings $f_n : X \to X$, with respective contraction constants α_n, for $n = 1, 2, \cdots, N$.

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- The notation used for IFS is $\{X; f_n, n = 1, 2, \cdots, N\}$ and its contraction factor is $\alpha = \max\{\alpha_n : n = 1, 2, \cdots, N\}$.
- The fixed point $P \in \mathcal{C}(X)$ of an IFS $\{X; f_n, n = 1, 2, \cdots, N\}$ is called **attractor** of IFS.

Iterated function system (IFS)

- The **Iterated function system** or **IFS** consists of a complete metric space (X, d) together with a finite set of contraction mappings $f_n : X \to X$, with respective contraction constants α_n, for $n = 1, 2, \cdots, N$.
- The notation used for IFS is $\{X; f_n, n = 1, 2, \cdots, N\}$ and its contraction factor is $\alpha = \max\{\alpha_n : n = 1, 2, \cdots, N\}$.
- The fixed point $P \in \mathcal{C}(X)$ of an IFS $\{X; f_n, n = 1, 2, \cdots, N\}$ is called **attractor** of IFS.
- Contractive affine transformations are commonly used for the construction of an **IFS**.

Theorem: Let $\{X; f_n, n = 1, 2, \cdots, N\}$ be **iterated function system** with contraction constant α. Then the transformation $F : \mathcal{C}(X) \to \mathcal{C}(X)$ defined by

$$F(A) = \cup_{n=1}^{N} f_n(A) \quad \text{for all } A \in \mathcal{C}(X)$$

is contraction mapping on the complete metric space $(\mathcal{C}(X), h(d))$ with contraction constant α. Hence it has a unique fixed point, $P \in \mathcal{C}(X)$

$$P = F(P) = \cup_{n=1}^{N} f_n(P).$$

Furthermore,

- for any $S \in \mathcal{C}(X)$, the sequence of compact sets $\{S, F(S), F^2(S), \cdots\}$ converges to fixed point of F.

Affine transformations

- A two dimensional **affine** transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form

$$f(x_1, x_2) = (ax_1 + bx_2 + e, cx_1 + dx_2 + f)$$

where a, b, c, d are real numbers. Equivalently,

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

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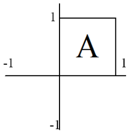
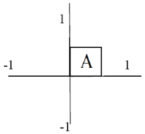
- The action of an affine transformation consists of **rotation, reflection, translation, shearing** and **scaling**.

- $f(x, y) = (x/2, y/2)$





Affine transformations

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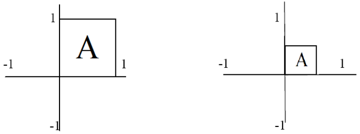
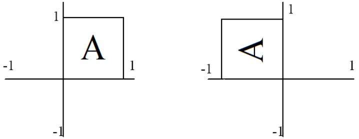


- $f(x,y) = (-y, x)$



Affine transformations

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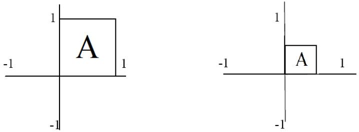

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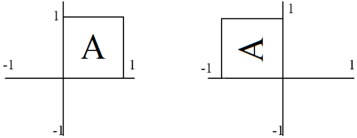
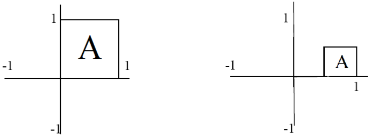


- $f(x, y) = (x/2, y/2) + (1/2, 0)$



Consider the IFS $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

$$F(A) = f_1(A) \cup f_2(A) \cup f_3(A) \quad \text{for all } A \in \mathcal{C}(\mathbb{R}^2).$$

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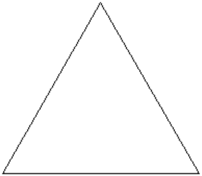

A

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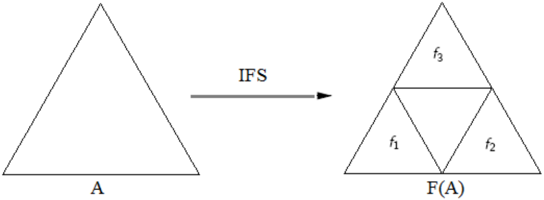

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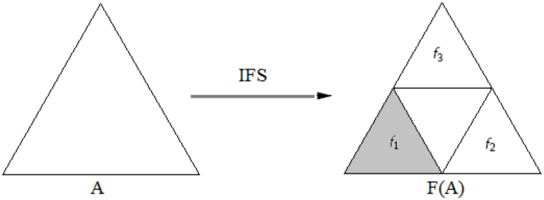

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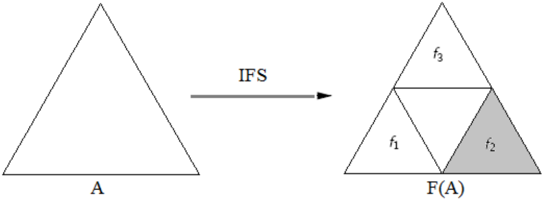

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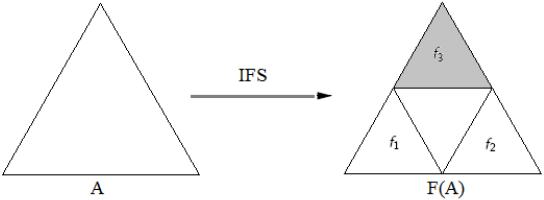

Sierpinski triangle

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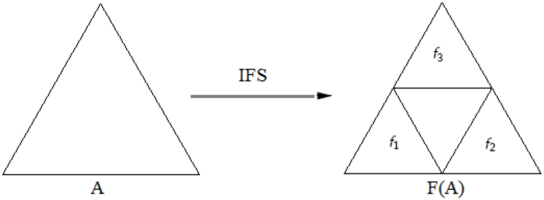

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

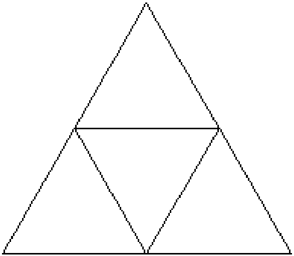

Figure: $F(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

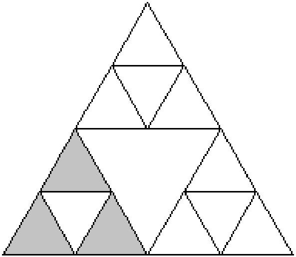

Figure: $F^2(A)$

Sierpinski triangle

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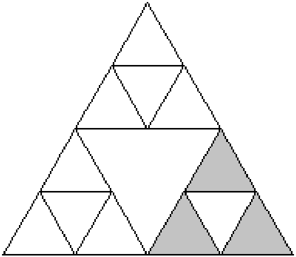

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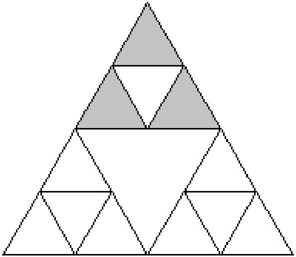

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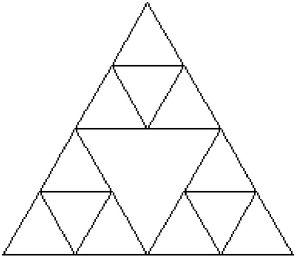

Figure: $F^2(A)$

Sierpinski triangle

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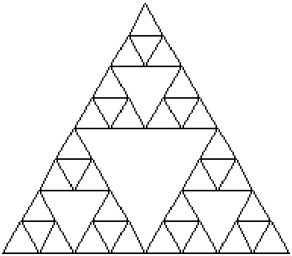

Figure: $F^3(A)$

Sierpinski triangle

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

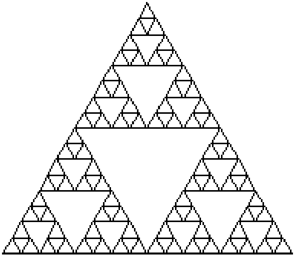

Figure: $F^4(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

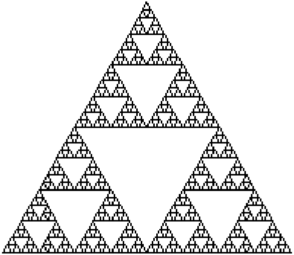

Figure: $F^5(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

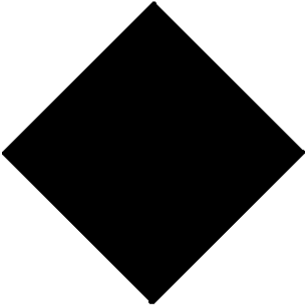

Figure: A

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

Figure: $F(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

Figure: $F^2(A)$

Sierpinski triangle

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

Figure: $F^3(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

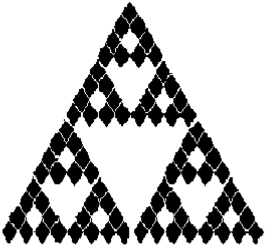

Figure: $F^4(A)$

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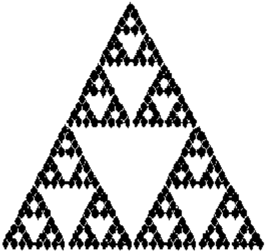

Figure: $F^5(A)$

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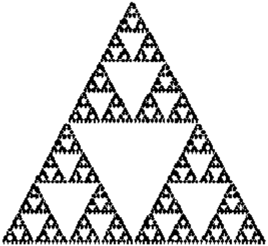
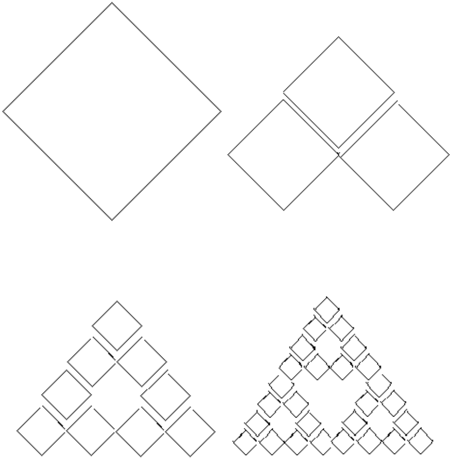


Figure: $F^6(A)$

IFS: $\{\mathbb{R}^2; f_1, f_2, f_3\}$



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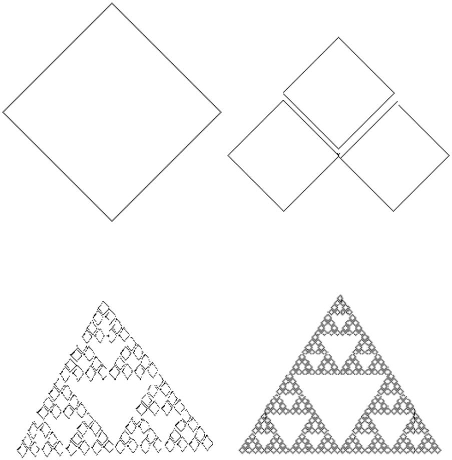

Consider the IFS: $\{\mathbb{R}^2; f_n, n = 1, \cdots, 7\}$ and $F : \mathcal{C}(\mathbb{R}^2) \rightarrow \mathcal{C}(\mathbb{R}^2)$

Snow flake

Consider the IFS: $\{\mathbb{R}^2; f_n, n = 1, \cdots, 7\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

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Snow flake

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4. $f_4(x, y) = (x/3, y/3) + (1/3, 1/\sqrt{3})$
5. $f_5(x, y) = (x/3, y/3) + (1, 1/\sqrt{3})$
6. $f_6(x, y) = (x/3, y/3) + (0, 2/\sqrt{3})$
7. $f_7(x, y) = (x/3, y/3) + (2/3, 2/\sqrt{3})$

$F(A) = f_1(A) \cup \cdots \cup f_7(A) \quad \text{for all } A \in \mathcal{C}(\mathbb{R}^2)$

Snow flake

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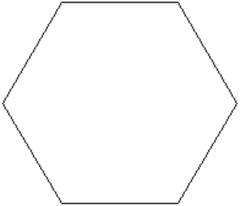
A



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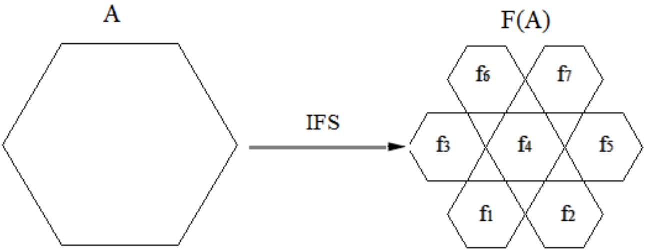

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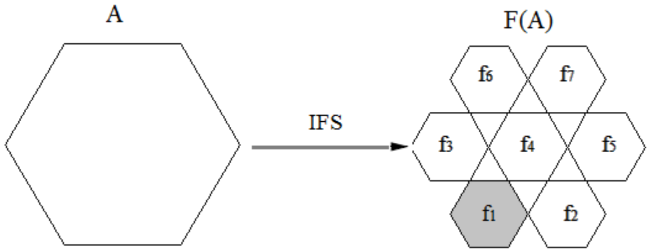

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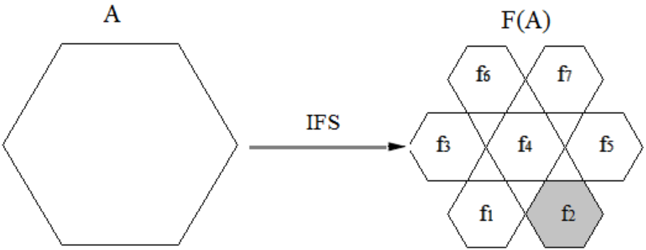

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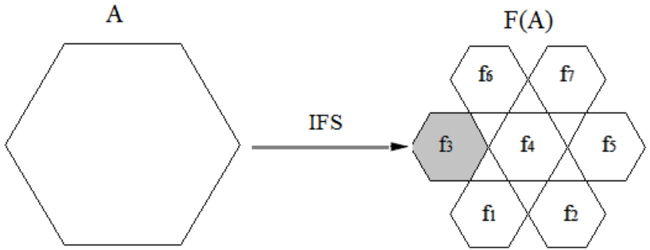

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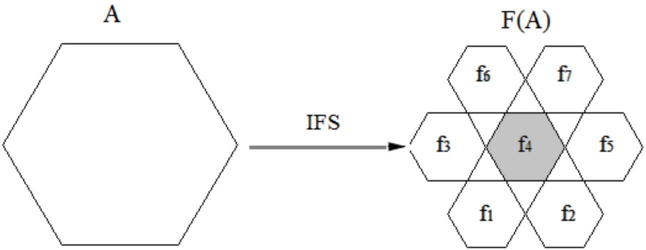

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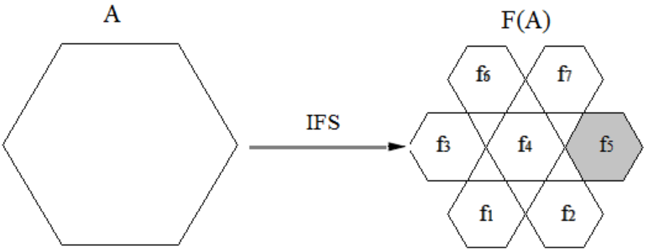

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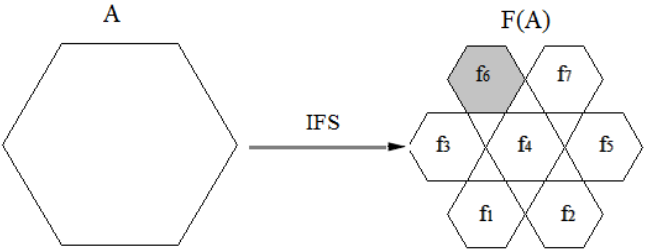

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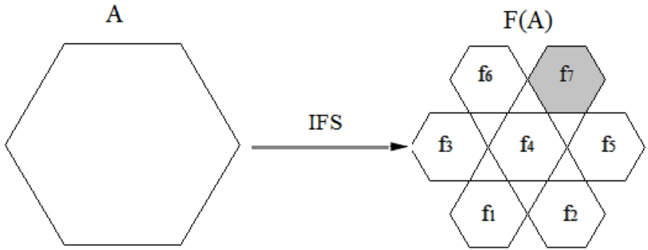

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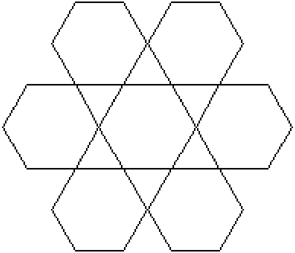

Figure: $F(A)$

Snow flake

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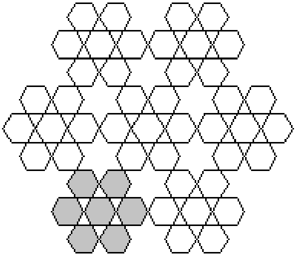

Figure: $F^2(A)$

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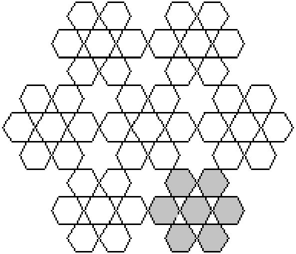

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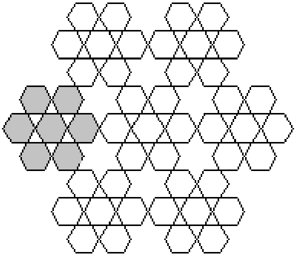

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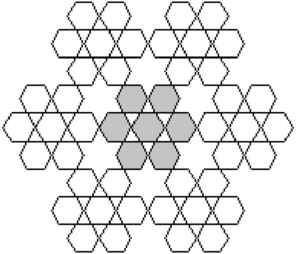

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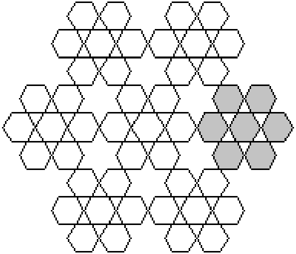

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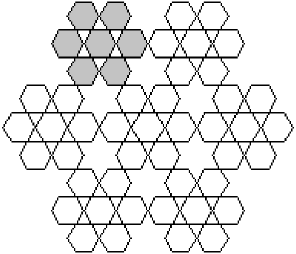

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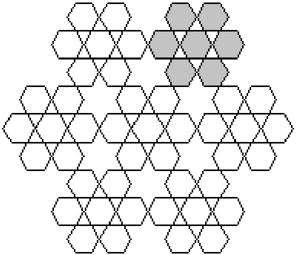

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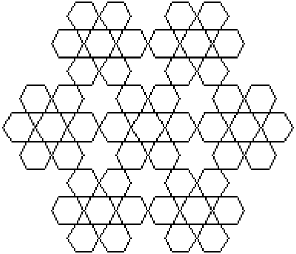

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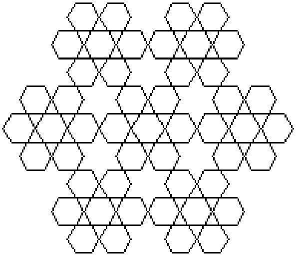

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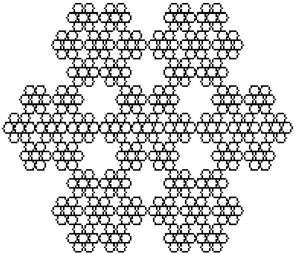

Figure: $F^3(A)$

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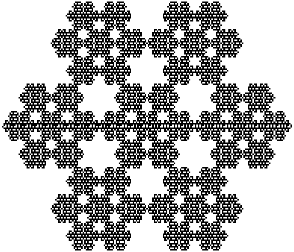

Figure: $F^4(A)$

Koch curve

Consider the IFS $\{\mathbb{R}^2; f_1, f_2, f_3, f_4\}$ and $F : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(\mathbb{R}^2)$

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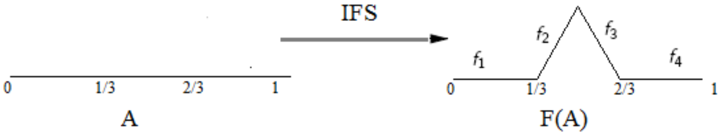

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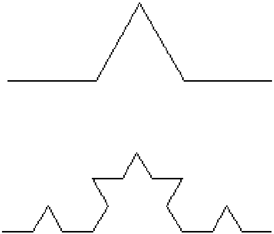

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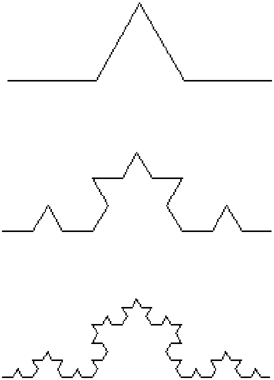

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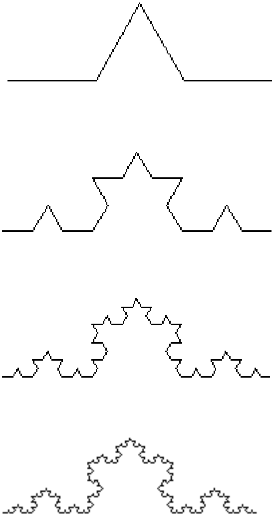

IFS with condensation

Definition: Let (X, d) be a metric space and let $C \in \mathcal{C}(X)$. Define a transformation $f_0 : \mathcal{C}(X) \to \mathcal{C}(X)$ by;

$$f_0(B) = C \quad \text{for all } B \in \mathcal{C}(X). \tag{5}$$

Then, f_0 is called **condensation** transformation and C is called the associated **condensation set**.

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- Condensation transformation $f_0 : \mathcal{C}(X) \to \mathcal{C}(X)$ is a contraction mapping on the metric space $(\mathcal{C}(X), h(d))$, with contraction constant $\alpha = 0$.

Definition: Let (X, d) be a metric space and let $C \in \mathcal{C}(X)$. Define a transformation $f_0 : \mathcal{C}(X) \to \mathcal{C}(X)$ by;

$$f_0(B) = C \quad \text{for all } B \in \mathcal{C}(X). \tag{5}$$

Then, f_0 is called **condensation** transformation and C is called the associated **condensation set**.

- Condensation transformation $f_0 : \mathcal{C}(X) \to \mathcal{C}(X)$ is a contraction mapping on the metric space $(\mathcal{C}(X), h(d))$, with contraction constant $\alpha = 0$.
- Hence, it has a unique fixed point, which is the condensation set itself.

Definition: Let $\{X; f_1, f_2, \cdots, f_n\}$ be an IFS with contraction constant $0 < \alpha < 1$. Let $f_0 : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ be a condensation transformation. Then $\{X; f_0, f_1, \cdots, f_n\}$ is called **iterated function system with condensation**, with contraction constant α.

IFS with condensation

Theorem: Let $\{X; f_n, n = 0, 1, \cdots, N\}$ be **iterated function system with condensation**, with contraction constant α. Then the transformation $F : \mathcal{C}(X) \to \mathcal{C}(X)$ defined by

$$F(A) = \cup_{n=0}^{N} f_n(A) \quad \text{for all } A \in \mathcal{C}(X)$$

is contraction mapping on the complete metric space $(\mathcal{C}(X), h(d))$ with contraction constant α. Hence it has a unique **fixed point**, $P \in \mathcal{C}(X)$

$$P = F(P) = \cup_{n=0}^{N} f_n(P).$$

Furthermore,

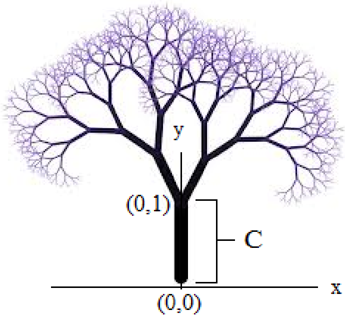
- for any $S \in \mathcal{C}(X)$, the sequence of compact sets $\{S, F(S), F^2(S), \cdots\}$ converges to **fixed point** of F.

Let $\{\mathbb{R}^2; f_0, f_1, f_2\}$ be the IFS with condensation,where

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2\cos\theta & -1/2\sin\theta \\ 1/2\sin\theta & 1/2\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2\cos\theta & 1/2\sin\theta \\ -1/2\sin\theta & 1/2\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let **condensation** set C be the trunk of the tree of unit height. That is $f_0(B) = C$ for all $B \in (\mathbb{R}^2)$. For $\theta = 60$ the attractor of above IFS with condensation set C is



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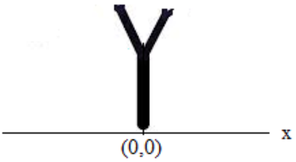

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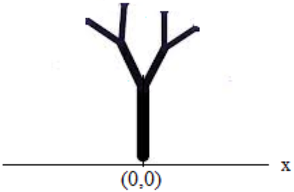

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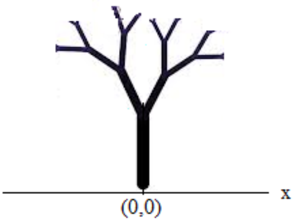

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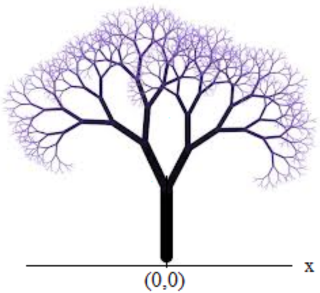
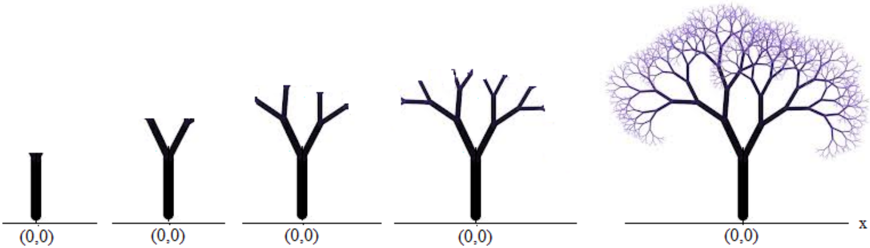


IFS with condensation



IFS with condensation

Changing the value of θ and condensation set C we have the following **attractors** to IFS with condensation.

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- Hence, **IFS** generates **fractals**.
- $(\mathcal{C}(X), h(d))$ is known as house for **fractals**.
- Same idea is led to partitioned iterated function system **PIFS** which is widely used for encoding of grey scale image. It is called **image compression** and is used to create special effects in gray scale image.

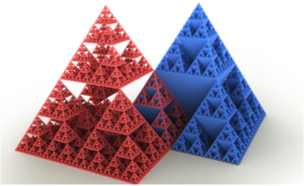
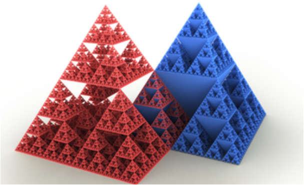


Figure: Sierpinski Pyramid

Fractals in nature



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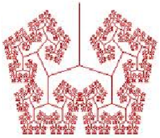

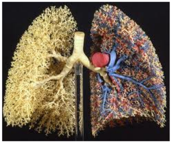

Figure: Human Lungs

Fractals in nature

Figure: Lightening fractal

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Figure: Sea shell

Fractals in nature

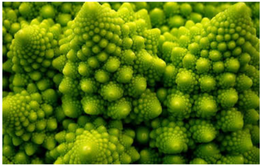

Figure: Fractals in plants

Fractals in nature

Figure: Fractals in plants

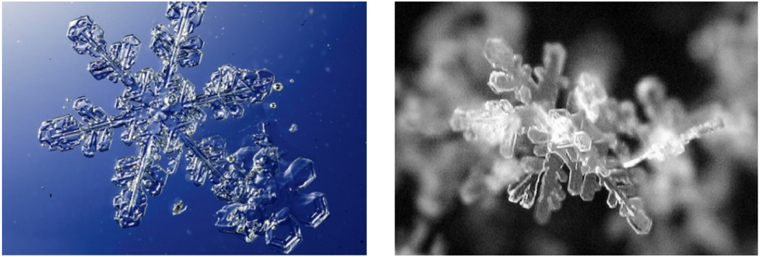

Figure: Snow flakes

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Thank you.