Bipartite Ramsey Number Pairs Involving Cycles

Ernst J. Joubert [1] and Johannes H. Hattingh [2]

[1]Department of Mathematics and Applied Mathematics,
University of Johannesburg, Auckland Park, 2006 SOUTH AFRICA

[2]Department of Mathematics and Statistics,
University of North Carolina Wilmington, NC 28403 USA

SAMS Conference,
University Of Stellenbosch,
2022

Basic Concepts

Definition 1:

Let $G = \{V, E\}$ be a graph with **vertex set** V and **edge set** E. The order of a graph (size, respectively) is denoted by **n** (**m**, respectively), where $\mathbf{n} = |V|$ and $\mathbf{m} = |E|$. Two vertices u and v are **adjacent** if there exists an edge between them.

Definition 2:

Let **X** and **Y** denote two disjoint vertex sets. We say that G is a **Bipartite graph** if no edge $e = xy$ of G is such that $x, y \in \mathbf{X}$ or $x, y \in \mathbf{Y}$. We denote $\mathcal{R}(G) = \mathbf{X}$ and $\mathcal{L}(G) = \mathbf{Y}$ as the right and left partite sets of G respectively. If $|\mathcal{R}(G)| = |\mathcal{L}(G)|$, then G is a **Balanced Bipartite graph**.

Basic Concepts

Definition 1:

Let $G = \{V, E\}$ be a graph with **vertex set** V and **edge set** E. The order of a graph (size, respectively) is denoted by **n** (**m**, respectively), where $\mathbf{n} = |V|$ and $\mathbf{m} = |E|$. Two vertices u and v are **adjacent** if there exists an edge between them.

Definition 2:

Let **X** and **Y** denote two disjoint vertex sets. We say that G is a **Bipartite graph** if no edge $e = xy$ of G is such that $x, y \in \mathbf{X}$ or $x, y \in \mathbf{Y}$. We denote $\mathcal{R}(G) = \mathbf{X}$ and $\mathcal{L}(G) = \mathbf{Y}$ as the right and left partite sets of G respectively. If $|\mathcal{R}(G)| = |\mathcal{L}(G)|$, then G is a **Balanced Bipartite graph**.

Basic Concepts

Definition 1:

Let $G = \{V, E\}$ be a graph with **vertex set** V and **edge set** E. The order of a graph (size, respectively) is denoted by **n** (**m**, respectively), where $\mathbf{n} = |V|$ and $\mathbf{m} = |E|$. Two vertices u and v are **adjacent** if there exists an edge between them.

Definition 2:

Let **X** and **Y** denote two disjoint vertex sets. We say that G is a **Bipartite graph** if no edge $e = xy$ of G is such that $x, y \in \mathbf{X}$ or $x, y \in \mathbf{Y}$. We denote $\mathcal{R}(G) = \mathbf{X}$ and $\mathcal{L}(G) = \mathbf{Y}$ as the right and left partite sets of G respectively. If $|\mathcal{R}(G)| = |\mathcal{L}(G)|$, then G is a **Balanced Bipartite graph**.

Basic Concepts

Definition 3:

For bipartite graphs $G_1, G_2, \ldots, G_k$, the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ is the least positive integer b so that any colouring of the edges of $K_{b,b}$ with k colours, will result in a copy of G_i, in the ith colour, for some i.

Definition 4:

If $G_i = G_j$ for all $i, j \in \{1, \ldots, k\}$, then the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ will be abbreviated as $b_k(G_1)$.

Basic Concepts

Definition 3:

For bipartite graphs $G_1, G_2, \ldots, G_k$, the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ is the least positive integer b so that any colouring of the edges of $K_{b,b}$ with k colours, will result in a copy of G_i, in the ith colour, for some i.

Definition 4:

If $G_i = G_j$ for all $i, j \in \{1, \ldots, k\}$, then the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ will be abbreviated as $b_k(G_1)$.

Basic Concepts

Definition 3:

For bipartite graphs $G_1, G_2, \ldots, G_k$, the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ is the least positive integer b so that any colouring of the edges of $K_{b,b}$ with k colours, will result in a copy of G_i, in the ith colour, for some i.

Definition 4:

If $G_i = G_j$ for all $i, j \in \{1, \ldots, k\}$, then the **bipartite Ramsey number** $b(G_1, G_2, \ldots, G_k)$ will be abbreviated as $b_k(G_1)$.

Background

- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper: **A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.**
- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Background

- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper:
  A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.

- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Background

- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper:
  A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.

- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Background

- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper:
  A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.

- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Background

- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper:
A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.

- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Background

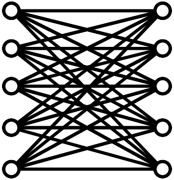
- The **existence** of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of **Erdös** and **Rado** in the paper: **A partition calculus in set theory, *Bull. Amer. Math. Soc* 62 (1956) 229–489.**

- The authors **J.H. Hattingh** and **M.A. Henning** considered the number $b(G_1, G_2, \ldots, G_k)$, where $G_i = C_4$, for all i. In particular, they showed the following:

Theorem 1:

(M. Henning and J. Hattingh) The bipartite Ramsey number $b_k(C_4)$ is at most $k^2 + k - 1$.

Illustration: $b_2(C_4)$



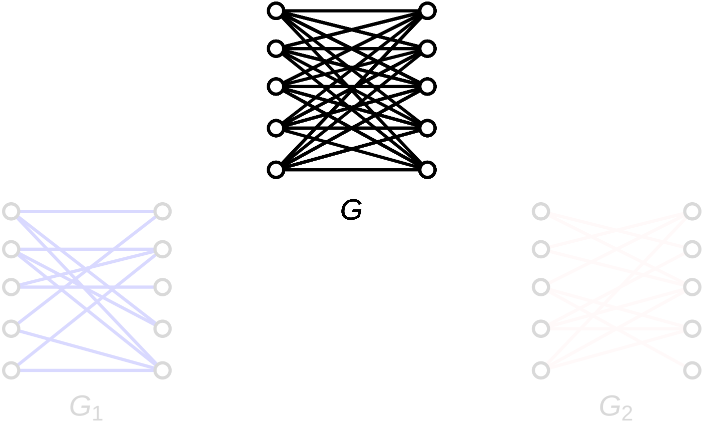
G

G_1

G_2

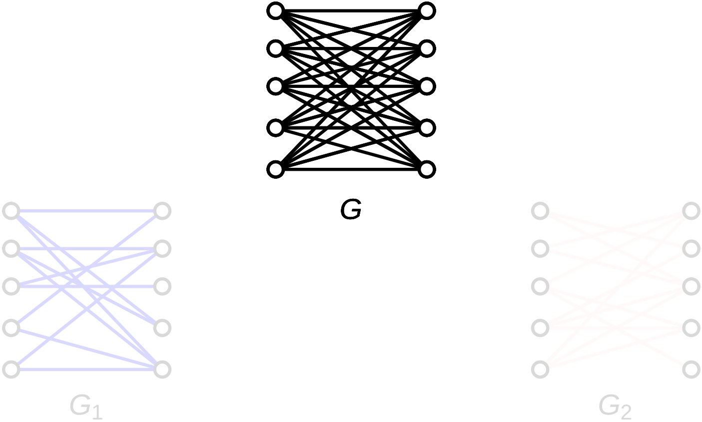
- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



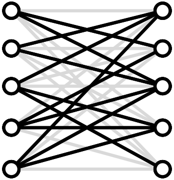
- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



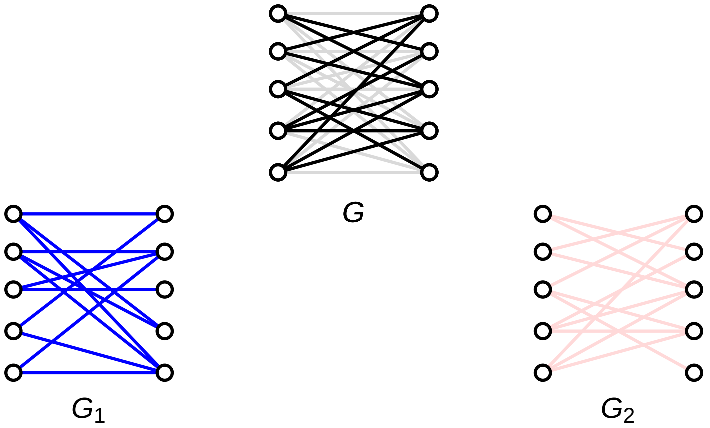
- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



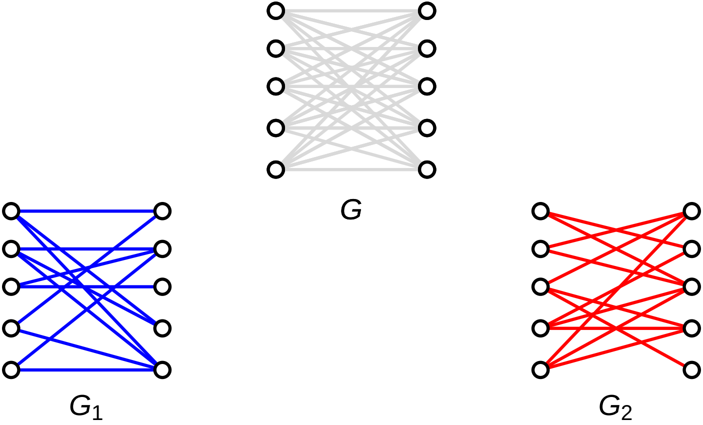
- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



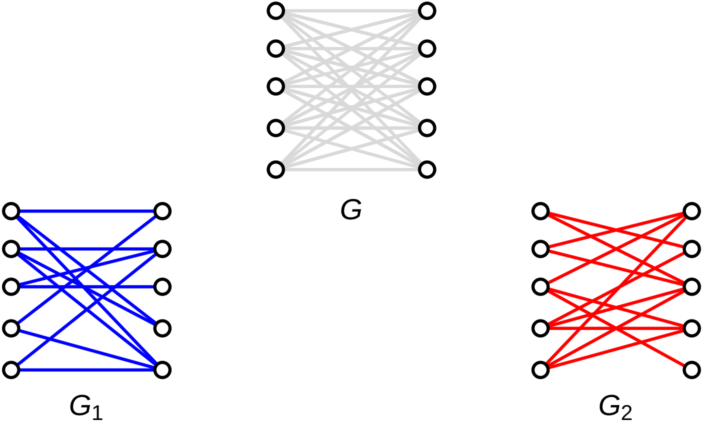
- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

Illustration: $b_2(C_4)$



- By Theorem 1, $b_2(C_4) \leq 5$. Colour the edges of a $K_{5,5}$ with two colours, say blue and red. The graph G_1 has a blue C_4.

- R. Zhang, Y. Sun and Y. Wu, The bipartite ramsey numbers $b(C_{2m}; C_{2n})$. *Int. Journ. of Math., Comp. Scien. and Engin.* **7** (2013), 42-45.
- Zhang, Sun and Wu conjectured that $b(C_{2s}) = 2s$, for $s \geq 3$.
- Zllong Yan and Yuejian Peng announced at the end of 2021 that they proved this conjecture. The paper has not been peer reviewed.
- Is this result optimal in the sense that it produces the smallest possible bipartite graph?

- R. Zhang, Y. Sun and Y. Wu, The bipartite ramsey numbers $b(C_{2m}; C_{2n})$. *Int. Journ. of Math., Comp. Scien. and Engin.* **7** (2013), 42-45.
- Zhang, Sun and Wu conjectured that $b(C_{2s}) = 2s$, for $s \geq 3$.
- Zllong Yan and Yuejian Peng announced at the end of 2021 that they proved this conjecture. The paper has not been peer reviewed.
- Is this result optimal in the sense that it produces the smallest possible bipartite graph?

- R. Zhang, Y. Sun and Y. Wu, The bipartite ramsey numbers $b(C_{2m}; C_{2n})$. *Int. Journ. of Math., Comp. Scien. and Engin.* **7** (2013), 42-45.
- Zhang, Sun and Wu conjectured that $b(C_{2s}) = 2s$, for $s \geq 3$.
- Zllong Yan and Yuejian Peng announced at the end of 2021 that they proved this conjecture. The paper has not been peer reviewed.
- Is this result optimal in the sense that it produces the smallest possible bipartite graph?

- R. Zhang, Y. Sun and Y. Wu, The bipartite ramsey numbers $b(C_{2m}; C_{2n})$. *Int. Journ. of Math., Comp. Scien. and Engin.* **7** (2013), 42-45.
- Zhang, Sun and Wu conjectured that $b(C_{2s}) = 2s$, for $s \geq 3$.
- ZIlong Yan and Yuejian Peng announced at the end of 2021 that they proved this conjecture. The paper has not been peer reviewed.
- Is this result optimal in the sense that it produces the smallest possible bipartite graph?

- R. Zhang, Y. Sun and Y. Wu, The bipartite ramsey numbers $b(C_{2m}; C_{2n})$. *Int. Journ. of Math., Comp. Scien. and Engin.* **7** (2013), 42-45.
- Zhang, Sun and Wu conjectured that $b(C_{2s}) = 2s$, for $s \geq 3$.
- ZIlong Yan and Yuejian Peng announced at the end of 2021 that they proved this conjecture. The paper has not been peer reviewed.
- Is this result optimal in the sense that it produces the smallest possible bipartite graph?

Definition 6:

Let a and b be positive integers with $a \geq b$. For bipartite graphs G_1 and G_2, the bipartite Ramsey number pair (a, b), denoted by $b_p(G_1, G_2) = (a, b)$, is an ordered pair of integers such that for any blue-red coloring of the edges of $K_{r,t}$, with $r \geq t$, either a blue copy of G_1 exists, or a red copy of G_2 exists, if and only if $r \geq a$ and $t \geq b$.

Faudree and Schelp determined bipartite Ramsey number pairs for paths. They showed that:

Definition 6:

Let a and b be positive integers with $a \geq b$. For bipartite graphs G_1 and G_2, the bipartite Ramsey number pair (a, b), denoted by $b_p(G_1, G_2) = (a, b)$, is an ordered pair of integers such that for any blue-red coloring of the edges of $K_{r,t}$, with $r \geq t$, either a blue copy of G_1 exists, or a red copy of G_2 exists, if and only if $r \geq a$ and $t \geq b$.

Faudree and Schelp determined bipartite Ramsey number pairs for paths. They showed that:

Definition 6:

Let a and b be positive integers with $a \geq b$. For bipartite graphs G_1 and G_2, the bipartite Ramsey number pair (a, b), denoted by $b_p(G_1, G_2) = (a, b)$, is an ordered pair of integers such that for any blue-red coloring of the edges of $K_{r,t}$, with $r \geq t$, either a blue copy of G_1 exists, or a red copy of G_2 exists, if and only if $r \geq a$ and $t \geq b$.

Faudree and Schelp determined bipartite Ramsey number pairs for paths. They showed that:

Theorem 2:

For positive integers n and m we have:

- $b_p(P_{2n}, P_{2m}) = (n+m-1, n+m-1)$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m, n+m-1)$ for $n \geq m-1$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m-1, n+m-1)$ for $n < m-1$,
- $b_p(P_{2n+1}, P_{2m+1}) = (n+m, n+m-1)$ for $n \neq m$,
- $b_p(P_{2n+1}, P_{2n+1}) = (2n+1, 2n-1)$.

Theorem 3:

(Henning, Joubert) For s sufficiently large, $b_p(C_{2s}, P_{2s}) = (2s-1, 2s-1)$.

Theorem 2:

For positive integers n and m we have:

- $b_p(P_{2n}, P_{2m}) = (n+m-1, n+m-1)$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m, n+m-1)$ for $n \geq m-1$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m-1, n+m-1)$ for $n < m-1$,
- $b_p(P_{2n+1}, P_{2m+1}) = (n+m, n+m-1)$ for $n \neq m$,
- $b_p(P_{2n+1}, P_{2n+1}) = (2n+1, 2n-1)$.

Theorem 3:

(Henning, Joubert) For s sufficiently large, $b_p(C_{2s}, P_{2s}) = (2s-1, 2s-1)$.

Theorem 2:

For positive integers n and m we have:

- $b_p(P_{2n}, P_{2m}) = (n+m-1, n+m-1)$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m, n+m-1)$ for $n \geq m-1$,
- $b_p(P_{2n+1}, P_{2m}) = (n+m-1, n+m-1)$ for $n < m-1$,
- $b_p(P_{2n+1}, P_{2m+1}) = (n+m, n+m-1)$ for $n \neq m$,
- $b_p(P_{2n+1}, P_{2n+1}) = (2n+1, 2n-1)$.

Theorem 3:

(Henning, Joubert) For s sufficiently large,
$b_p(C_{2s}, P_{2s}) = (2s-1, 2s-1)$.

Lemma

Let $s \geq 18$ be an integer. If a blue-red coloring of the edges of $K_{2s,2s-1}$ results in a red copy of C_{2s-2}, then there either exists a red copy of C_{2s} or a blue copy of C_{2s}.

Theorem 4:

If s is an integer such that $s \geq \max\{18, (b(C_{34}, C_{34}) + 1)/2\}$, then $b_p(C_{2s}) = (2s, 2s - 1)$.

Lemma

Let $s \geq 18$ be an integer. If a blue-red coloring of the edges of $K_{2s,2s-1}$ results in a red copy of C_{2s-2}, then there either exists a red copy of C_{2s} or a blue copy of C_{2s}.

Theorem 4:

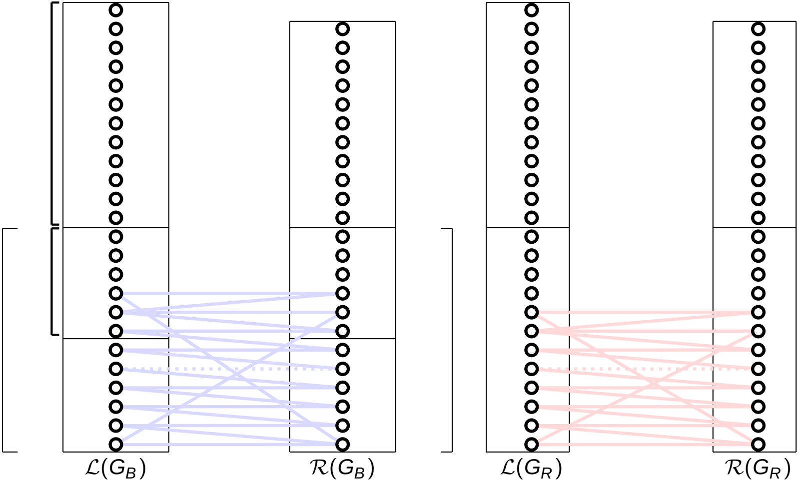
If s is an integer such that $s \geq \max\{18, (b(C_{34}, C_{34}) + 1)/2\}$, then $b_p(C_{2s}) = (2s, 2s - 1)$.

Lemma

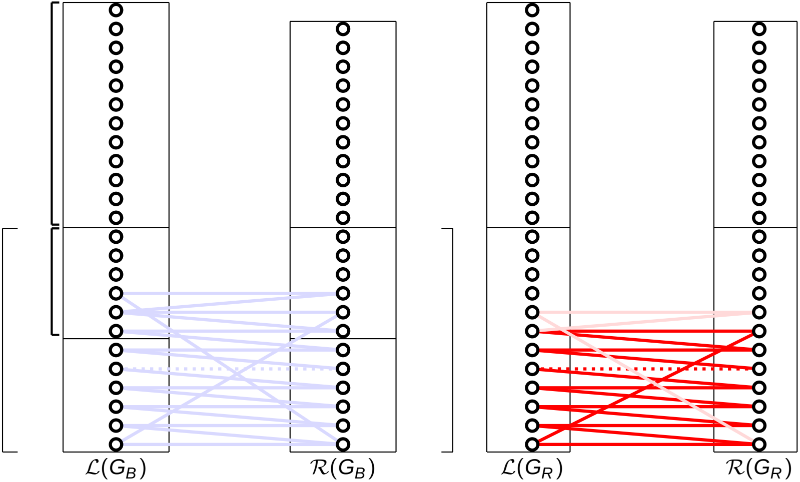
Let $s \geq 18$ be an integer. If a blue-red coloring of the edges of $K_{2s,2s-1}$ results in a red copy of C_{2s-2}, then there either exists a red copy of C_{2s} or a blue copy of C_{2s}.

Theorem 4:

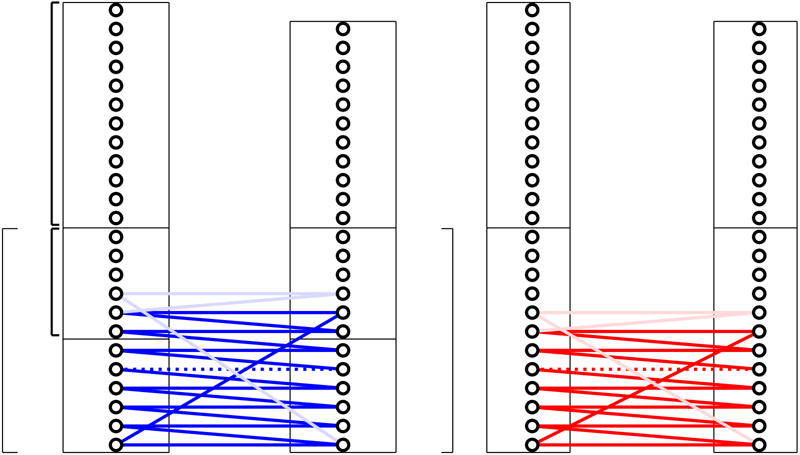
If s is an integer such that $s \geq \max\{18, (b(C_{34}, C_{34}) + 1)/2\}$, then $b_p(C_{2s}) = (2s, 2s - 1)$.



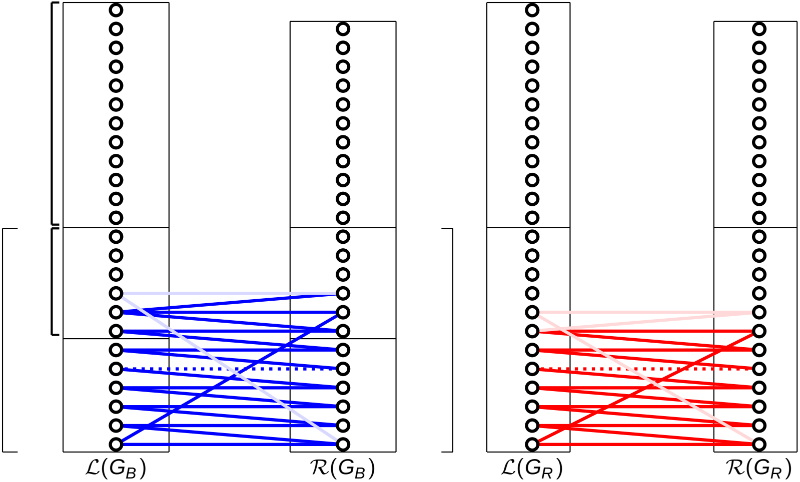
$2s-1 \geq b(C_{34}, C_{34}) \implies G_R \implies C_{2q} \implies 17 \leq q \leq s-1 \implies G_B$ has a $C_{2(q+1)} \implies 18 \leq q+1 \leq s \implies$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$ $\implies G_R$ has a $C_{2q'+2}$



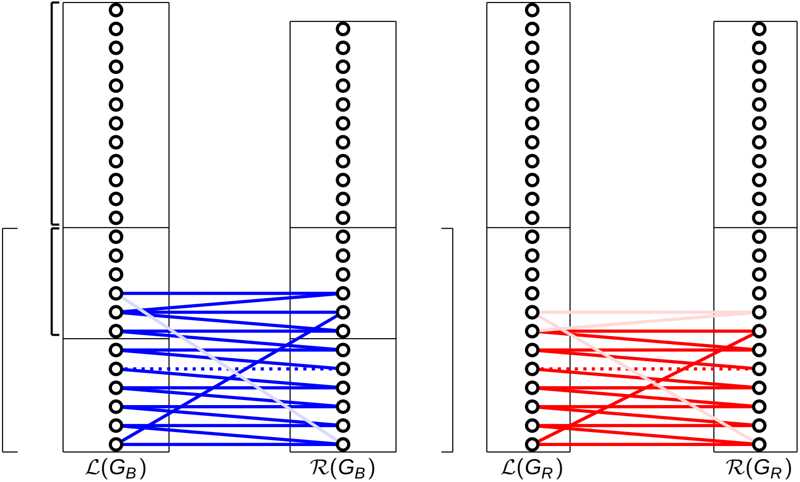
$2s - 1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s - 1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q + 1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s - 1$ $\Longrightarrow G_R$ has a $C_{2q'+2}$



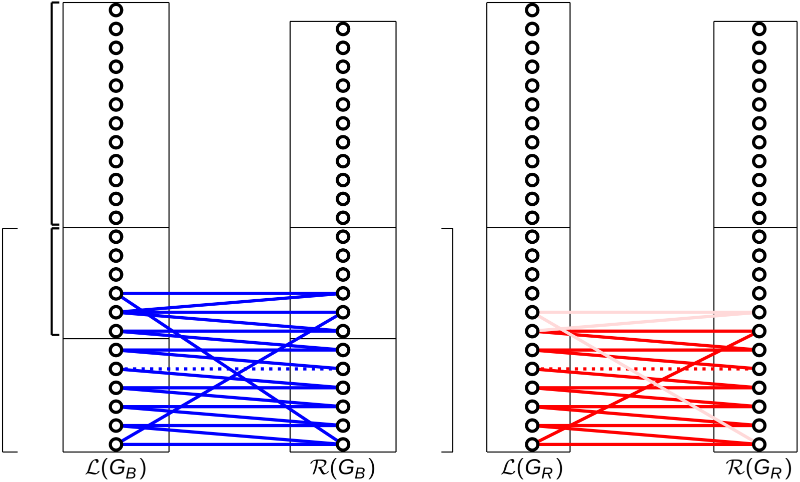
$2s - 1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s - 1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q + 1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s - 1$ $\Longrightarrow G_R$ has a $C_{2q'+2}$



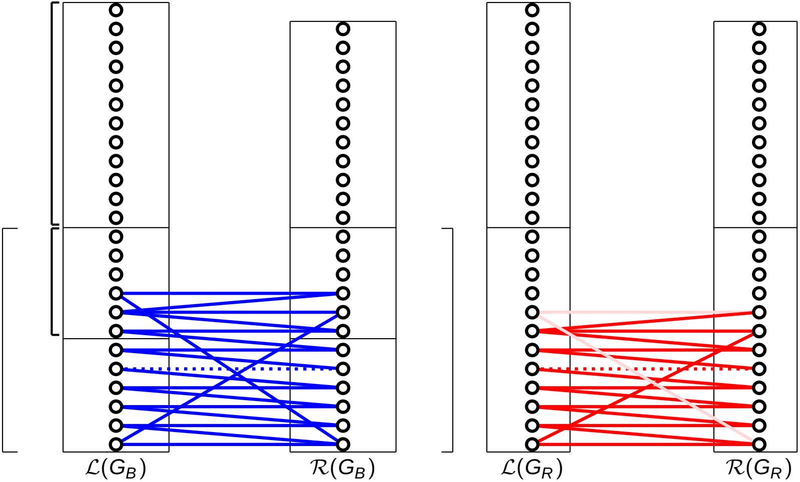
$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$ $\Longrightarrow G_R$ has a $C_{2q'+2}$



$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$ $\Longrightarrow G_R$ has a $C_{2q'+2}$

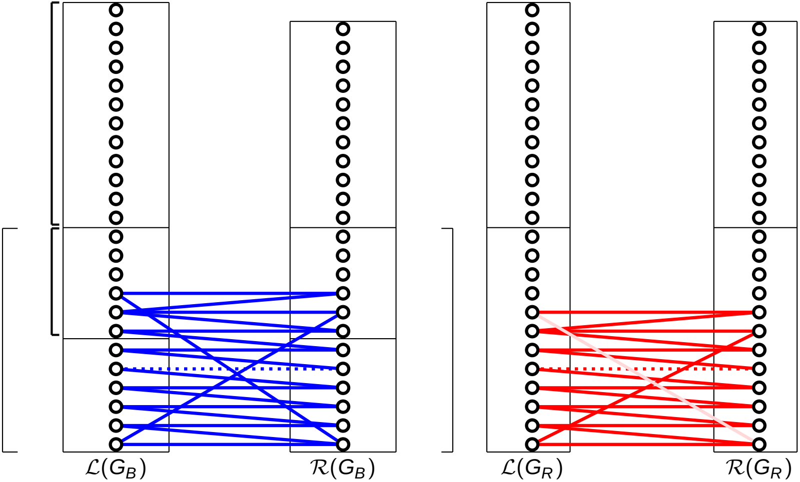


$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$
$\Longrightarrow G_R$ has a $C_{2q'+2}$

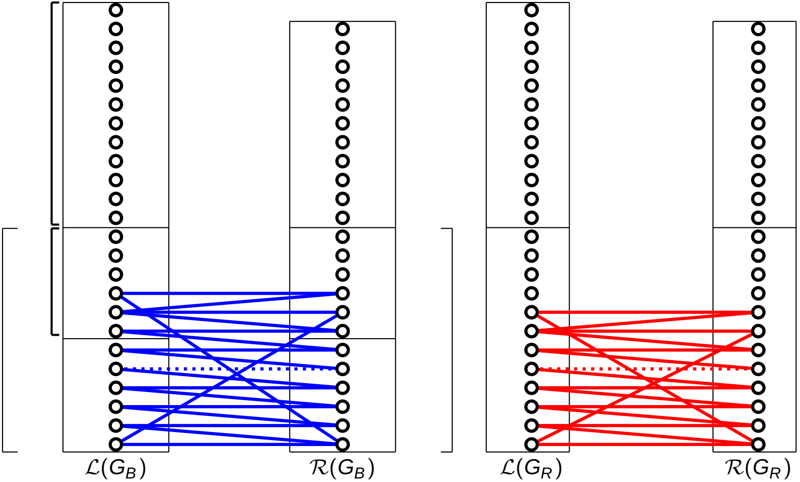


$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$

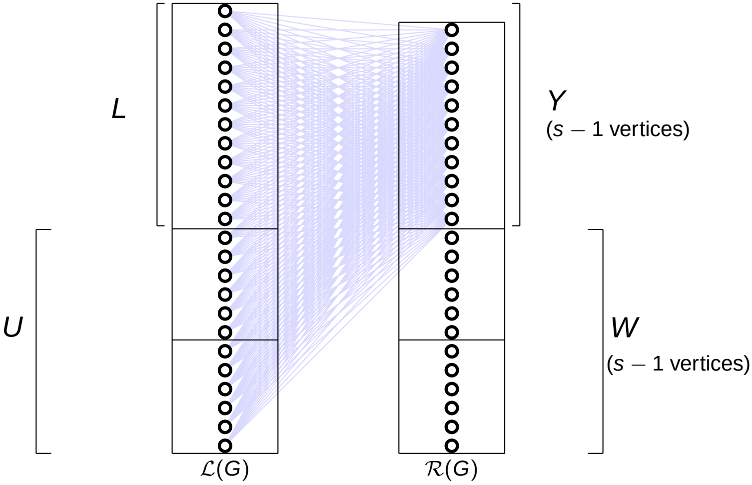
$\Longrightarrow G_R$ has a $C_{2q'+2}$



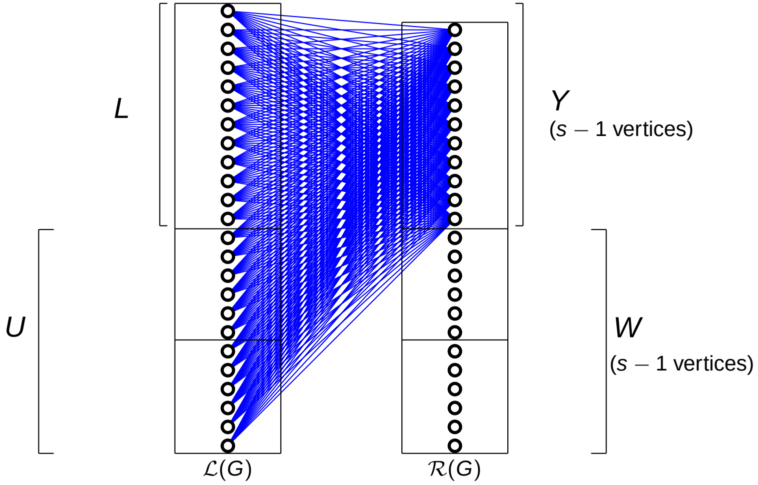
$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$
$\Longrightarrow G_R$ has a $C_{2q'+2}$



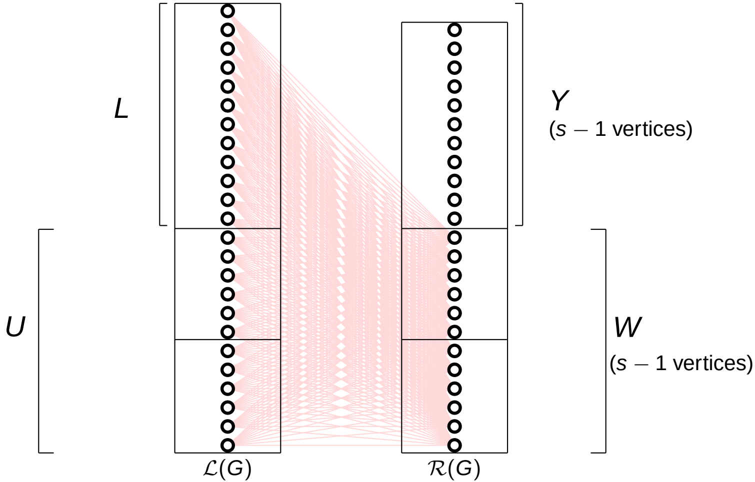
$2s-1 \geq b(C_{34}, C_{34}) \Longrightarrow G_R \Longrightarrow C_{2q} \Longrightarrow 17 \leq q \leq s-1 \Longrightarrow G_B$ has a $C_{2(q+1)} \Longrightarrow 18 \leq q+1 \leq s \Longrightarrow$ Pick $C_{2q'}$ in G_B, where $19 \leq q < q' \leq s-1$ $\Longrightarrow G_R$ has a $C_{2q'+2}$



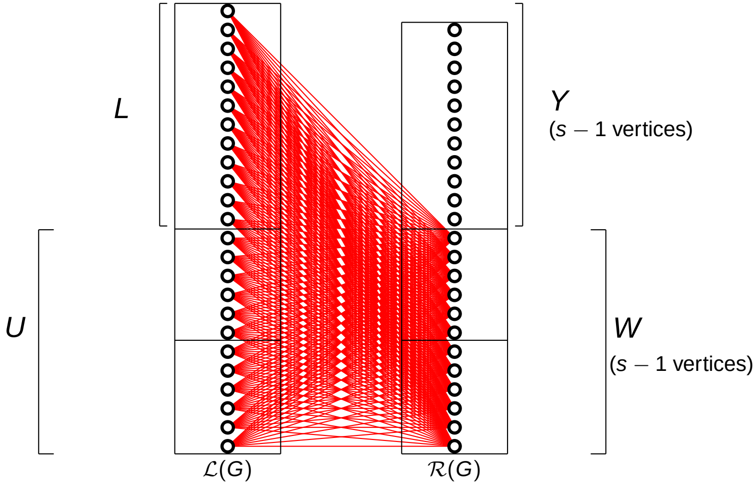
Lower bound coloring Part 1.



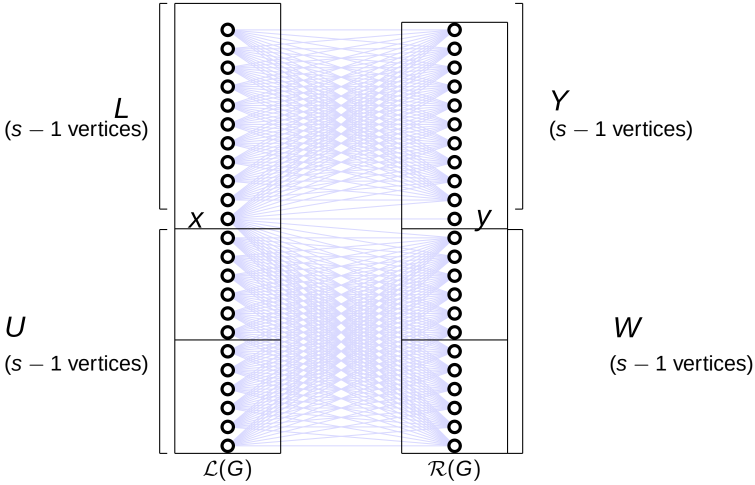
Lower bound coloring Part 1.



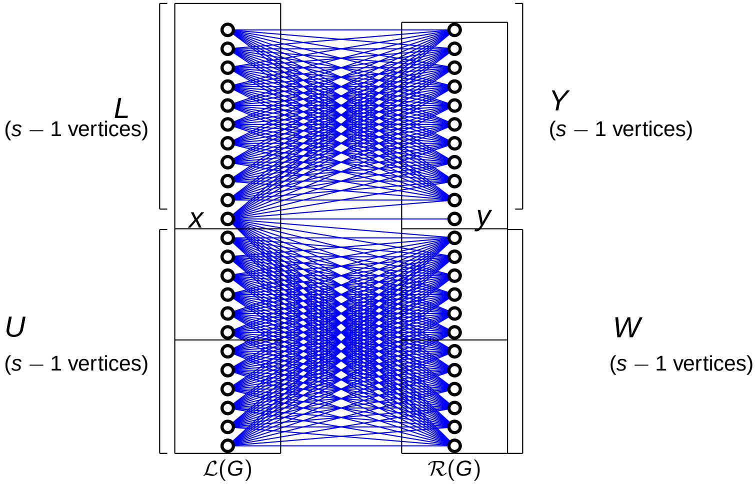
Lower bound coloring Part 1.



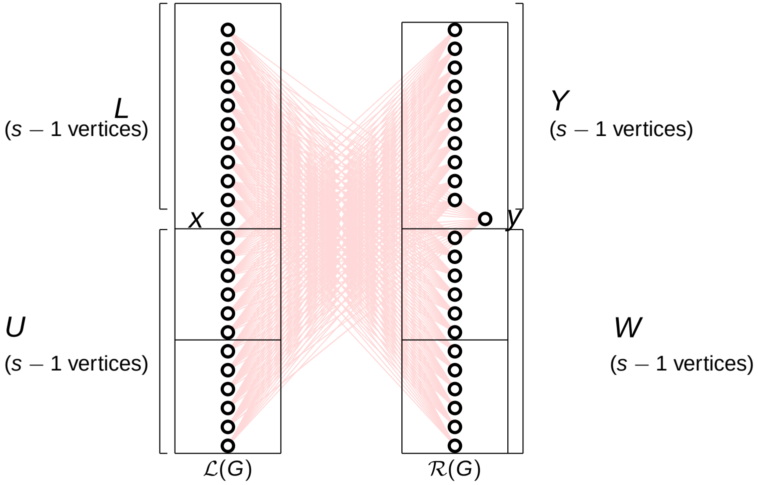
Lower bound coloring Part 1.



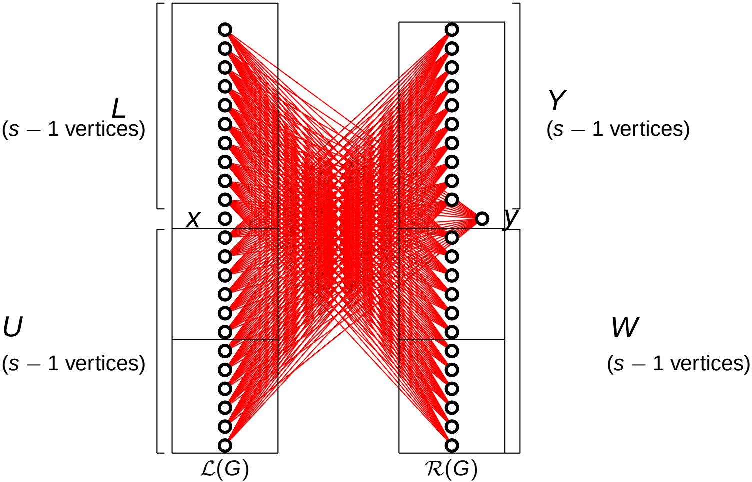
Lower bound coloring Part 2.



Lower bound coloring Part 2.



Lower bound coloring Part 2.

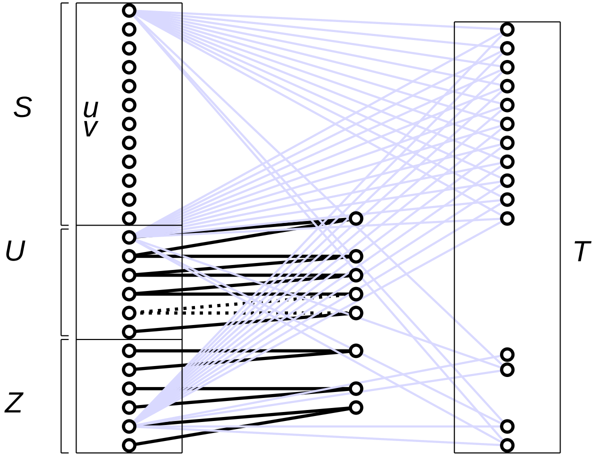


Lower bound coloring Part 2.

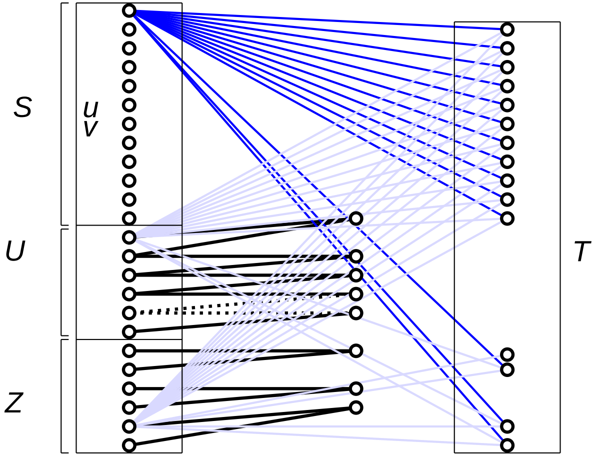
Lemma 2:

Let $u, v \in S$, $u', v' \in S - \{u, v\}$ and $|T| \geq 4$. For the graph G, the following holds:

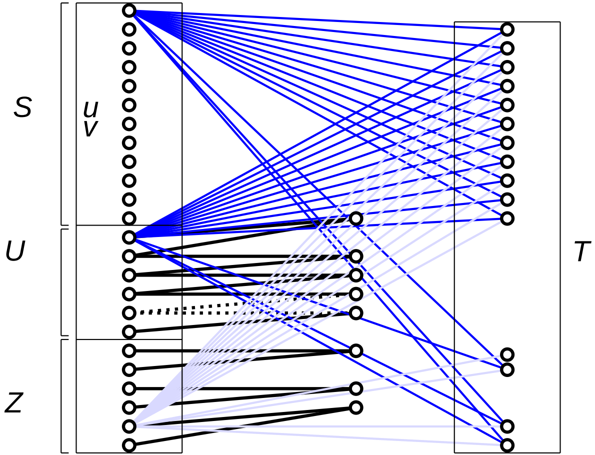
- If $|T| - 1 \geq k + 1$, $|S| + k \geq |T|$, and every vertex in S has at least $|T| - 1$ neighbors in T, then for the set Z_1, there exists a $u - v$ path on $2|T| - 1 + 2k$ vertices.
- If $|S| \geq |T| - 1$, and every vertex in S has at least $|T| - 1$ neighbors in T, then there exists a cycle on $2|T| - 2$ vertices.
- If $|Z_1| = 2$ ($Z_2 \neq \emptyset$, repsectively), $|S| + 2 \geq |T| - 1$ ($|S| + 1 \geq |T| - 1$, resp.), $|T| \geq 6$, and every vertex in S has at least $|T| - 1$ neighbors in T, then there exists a cycle on $2|T| + 2$ vertices.
- If $|T| \geq k + 3$, $|S| + k - 1 \geq |T|$, every vertex in S has at least $|T| - 1$ neighbors in T, and u' and v' both have $|T|$ neighbors in T, then for the set Z_1 there exists a $u - v$ path on $2|T| + 1 + 2k$ vertices.
- If $|T| - 1 \geq k + 2$, $|S| + k + 1 \geq |T|$ and every vertex in S has at least $|T| - 1$ neighbors in T, then for the set Z_1 and the path P', there exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.
- If $|T| \geq k + 4$, $|S| + k + 1 \geq |T| + 1$, every vertex in S has at least $|T| - 1$ neighbors in T, and u' and v' both have $|T|$ neighbors in T, then for the set Z_1 and the path P' there exists a $u - v$ path on $2|T| + 2k + \ell$ vertices.
- If $|T| \geq 8$, $|S| + k \geq |T| - 2$, $k \in \{0, 1\}$, and, if $k = 1$, the vertices z_1 and z_1' are joined to at least $|T| - 2$ neighbors in T, then there exists a $u - v$ path on $2|T| - 5 + 2k$ vertices.



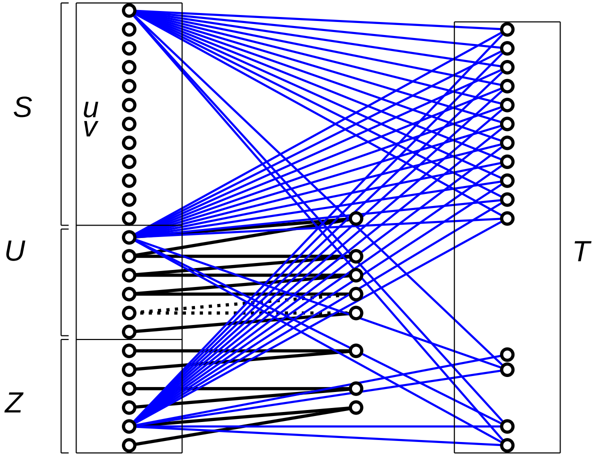
Construction illustration.



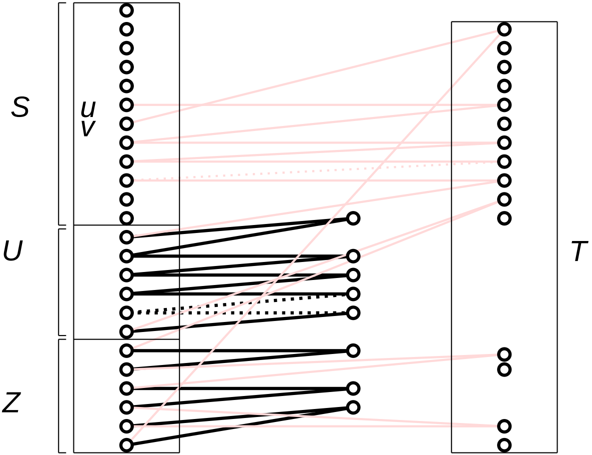
Construction illustration.



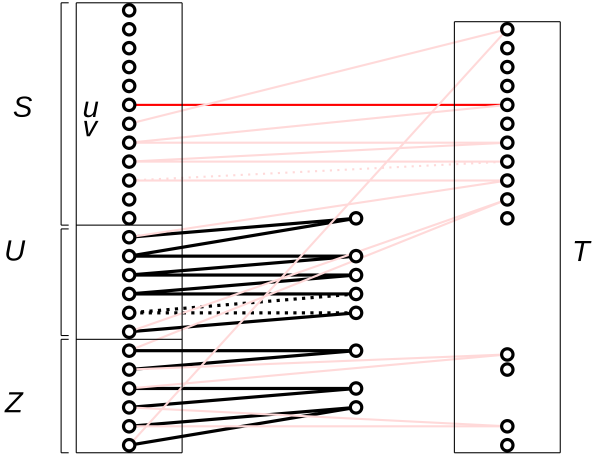
Construction illustration.



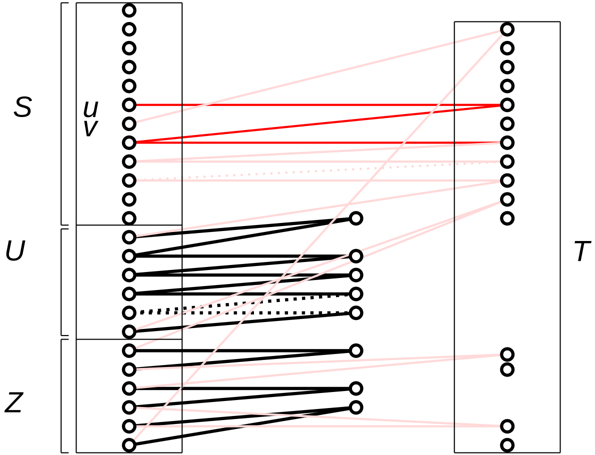
Construction illustration.



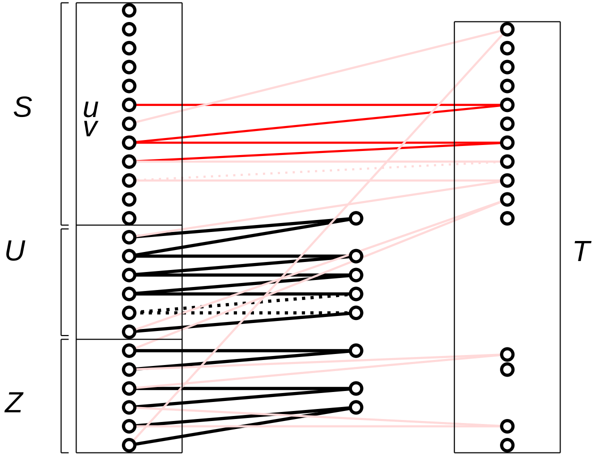
There exists a $u-v$ path on $2|T|-1+2k+\ell-1$ vertices.



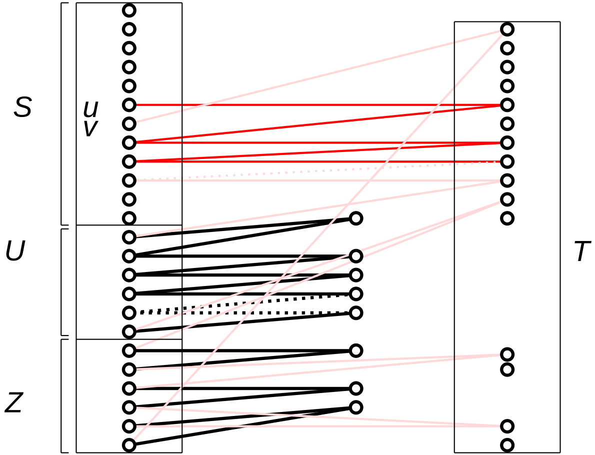
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



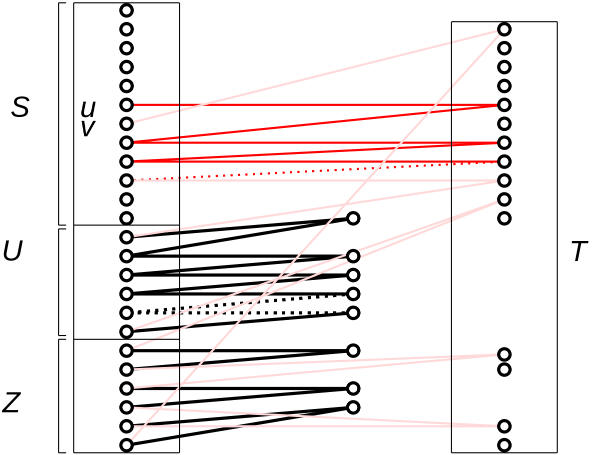
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



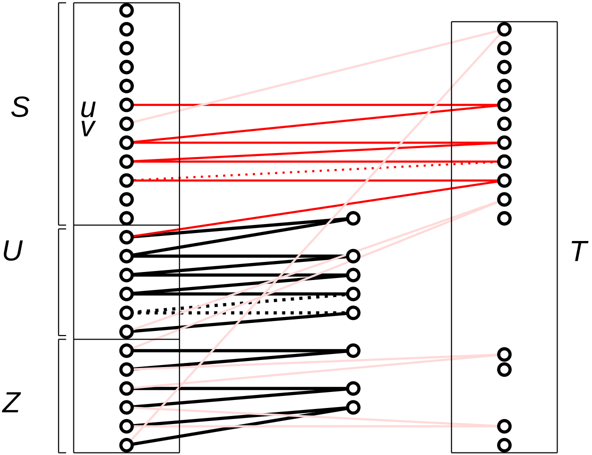
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



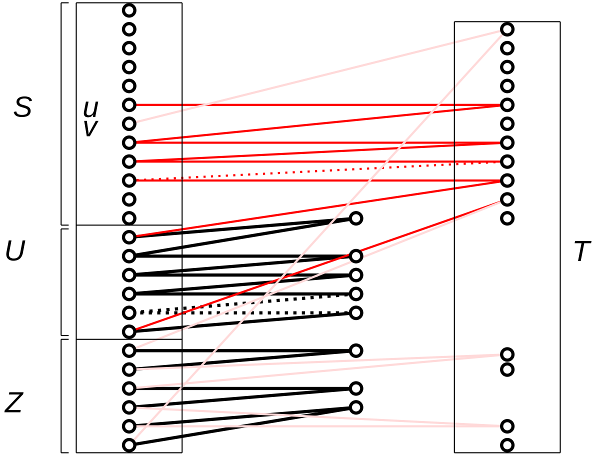
There exists a $u - v$ path on $2\,|T| - 1 + 2k + \ell - 1$ vertices.



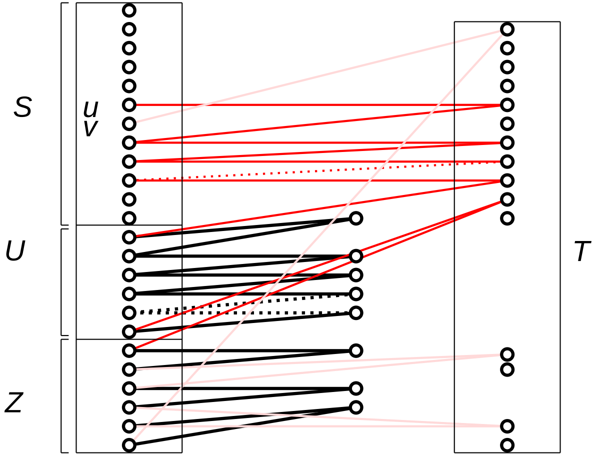
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



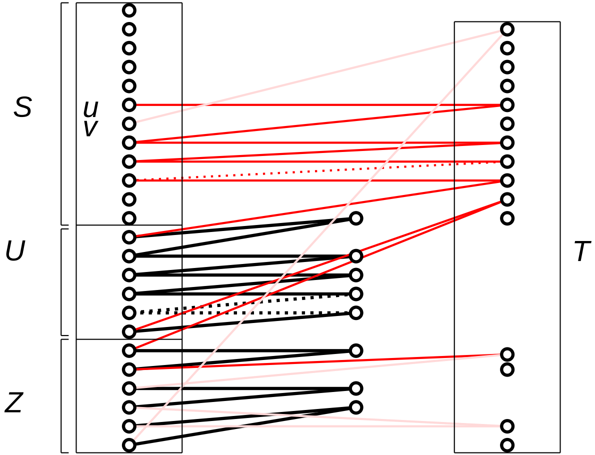
There exists a $u - v$ path on $2\,|T| - 1 + 2k + \ell - 1$ vertices.



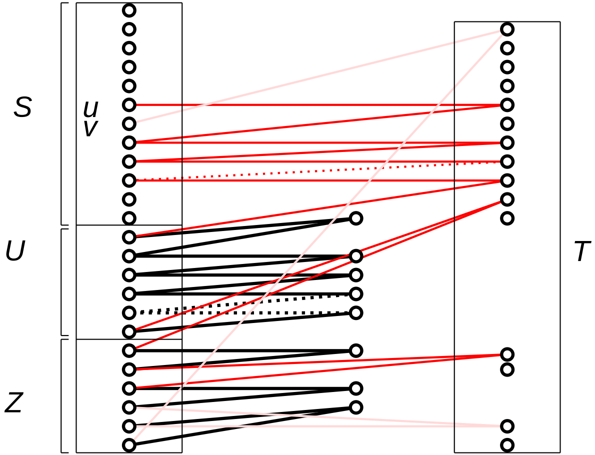
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



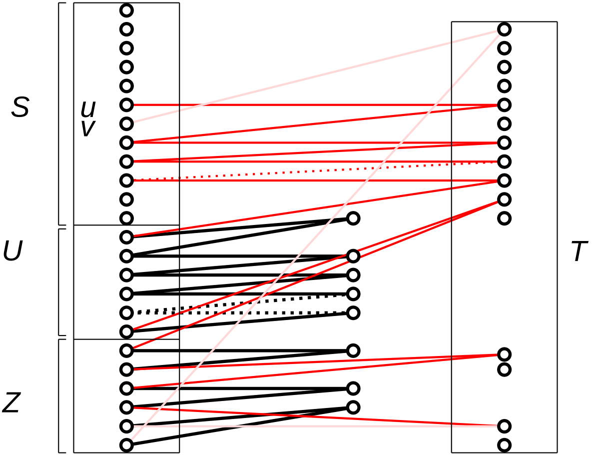
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



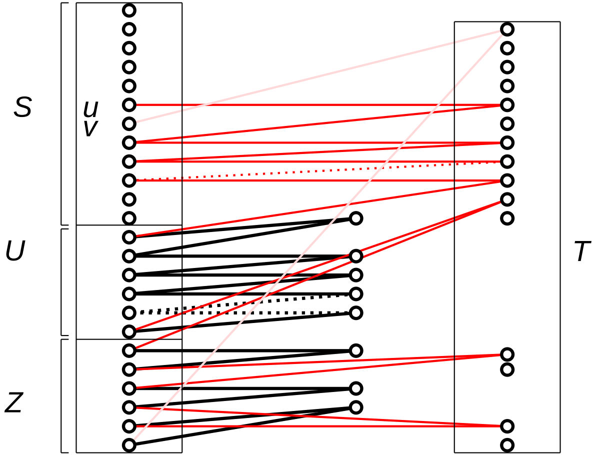
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



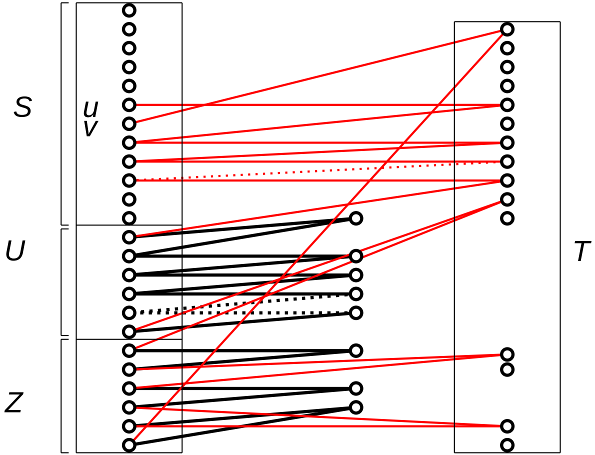
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.



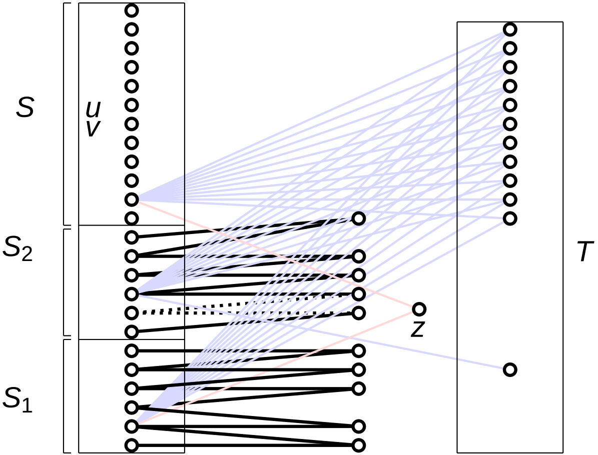
There exists a $u-v$ path on $2|T|-1+2k+\ell-1$ vertices.



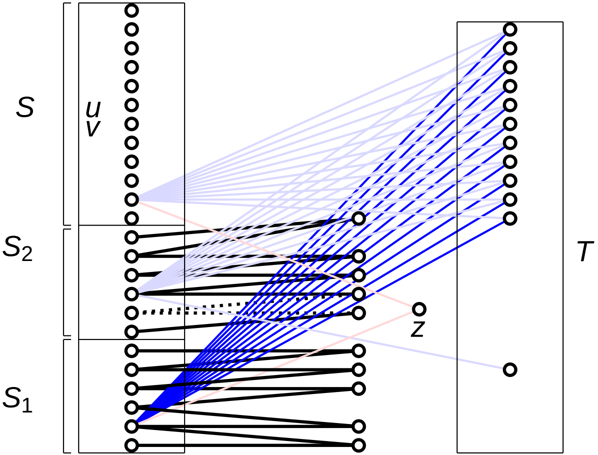
There exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.

Lemma 3:

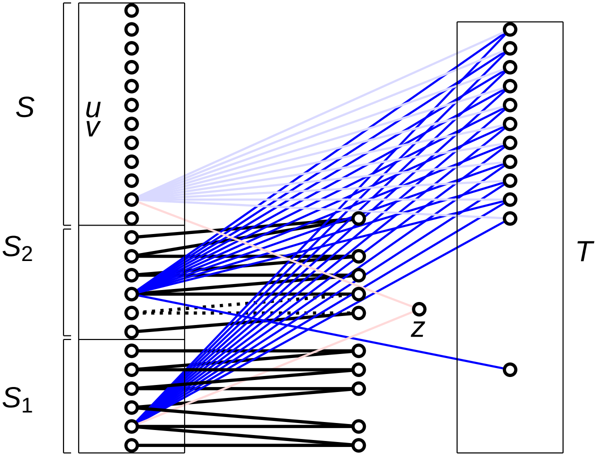
Let $u, v \in S$ and $|T| \geq 3$. There exists a $u - v$ path, that alternates between the sets $S \cup S_1 \cup S_2$ and $T \cup T_1 \cup T_2 \cup \{z\}$, on $2|T| - 1 + k + \ell$ vertices.



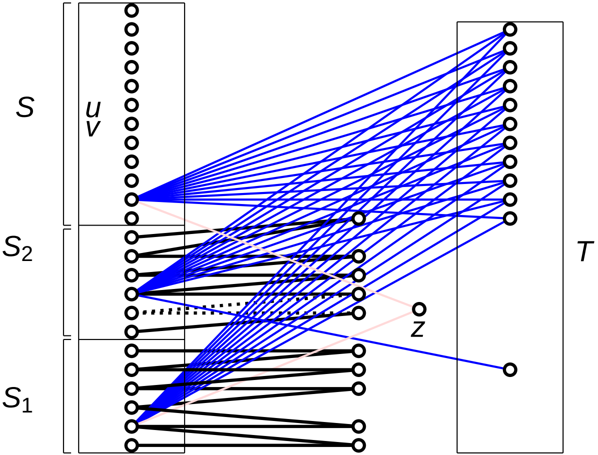
Construction illustration.



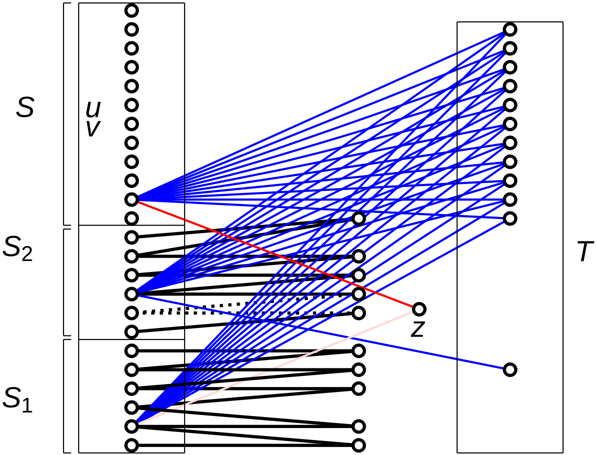
Construction illustration.



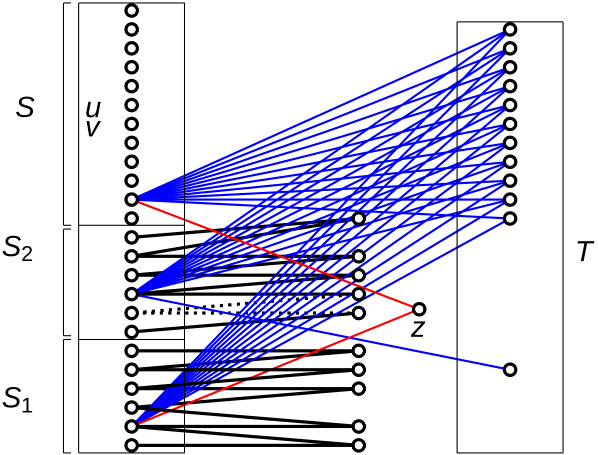
Construction illustration.



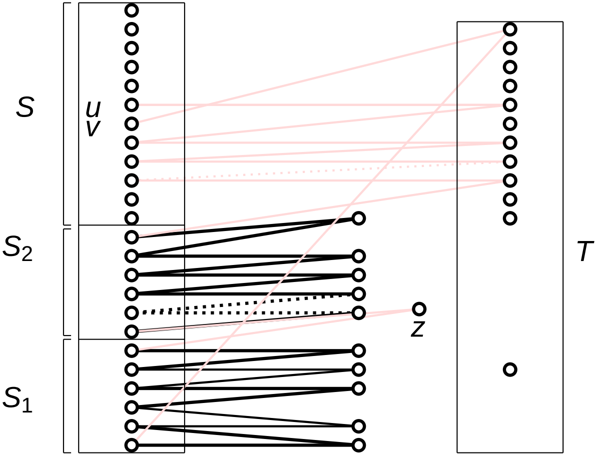
Construction illustration.



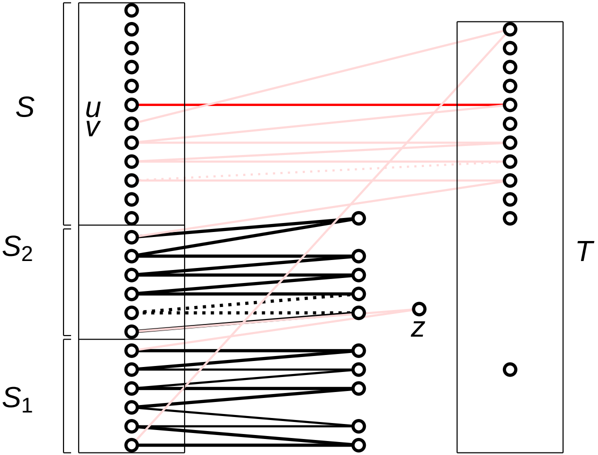
Construction illustration.



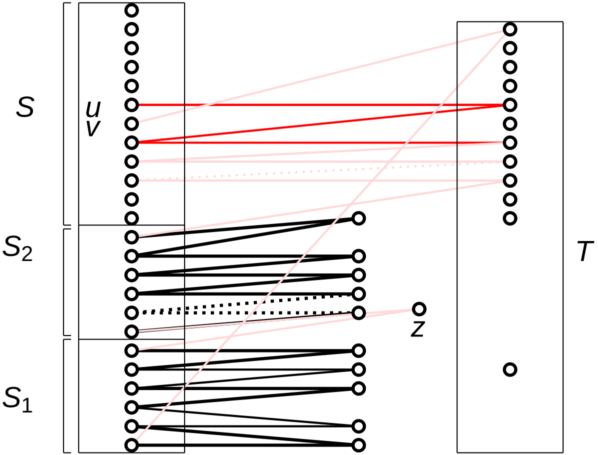
Construction illustration.



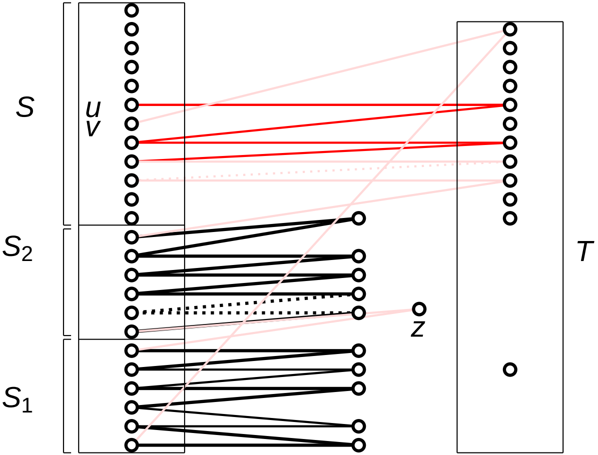
There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



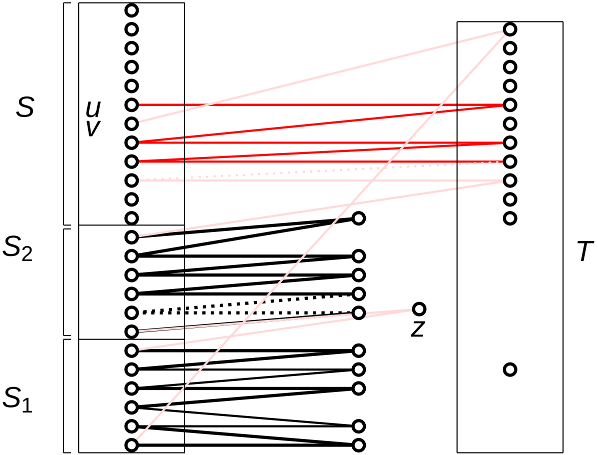
There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



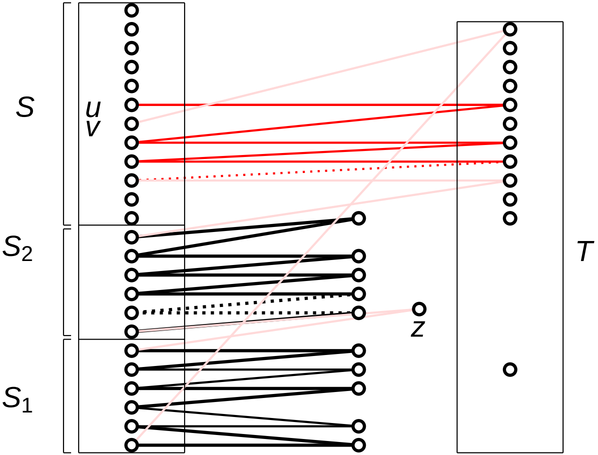
There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



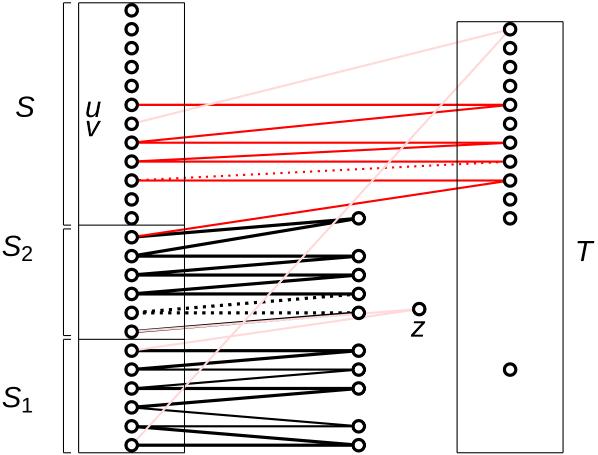
There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



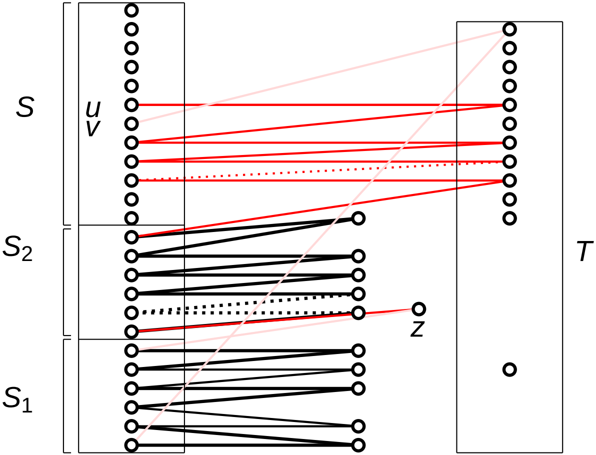
There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



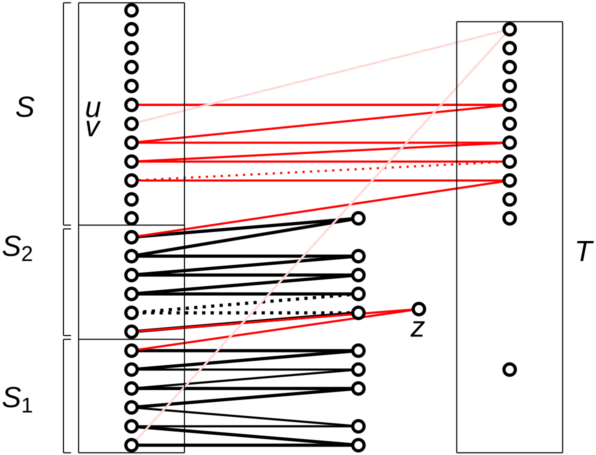
There exists a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices.



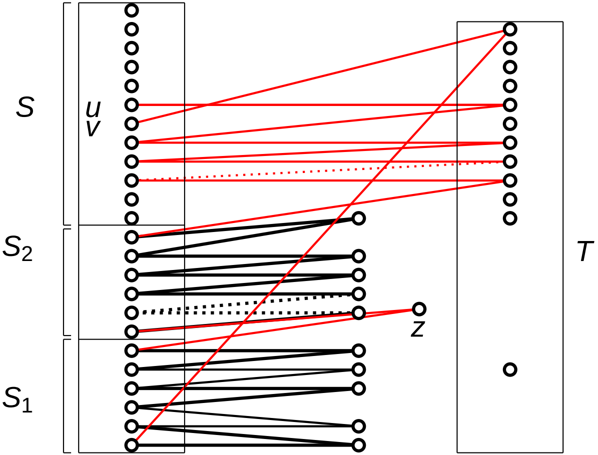
There exists a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices.



There exists a $u-v$ path on $2|T|-1+k+\ell$ vertices.



There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.



There exists a $u - v$ path on $2|T| - 1 + k + \ell$ vertices.

Lemma 4:

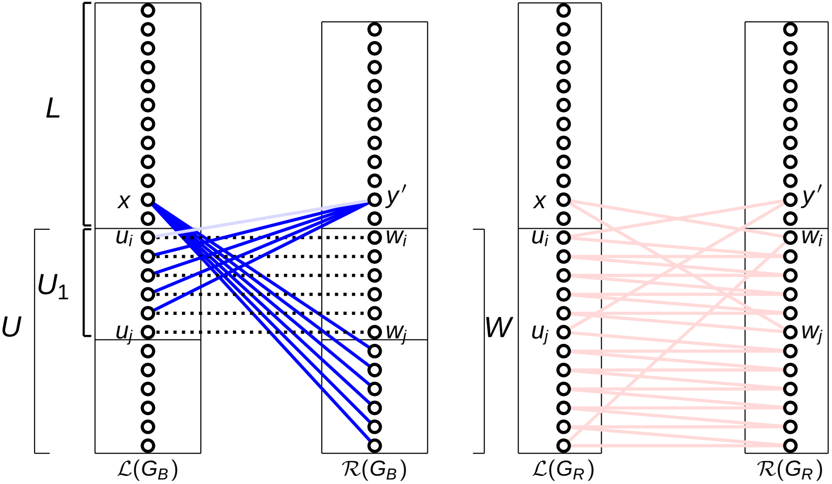
Let S' and T' be the partite sets of a bipartite graph, where, for some integer $k' \geq 0$, $|S'| - k' > |T'|$. If each vertex in S' has more that k' neighbors in T', then there exists $k' + 1$ disjoint copies of $K_{1,2}$, where each copy has its central vertex (two end vertices, resp.) in T' (S', resp.).

Lemma 5:

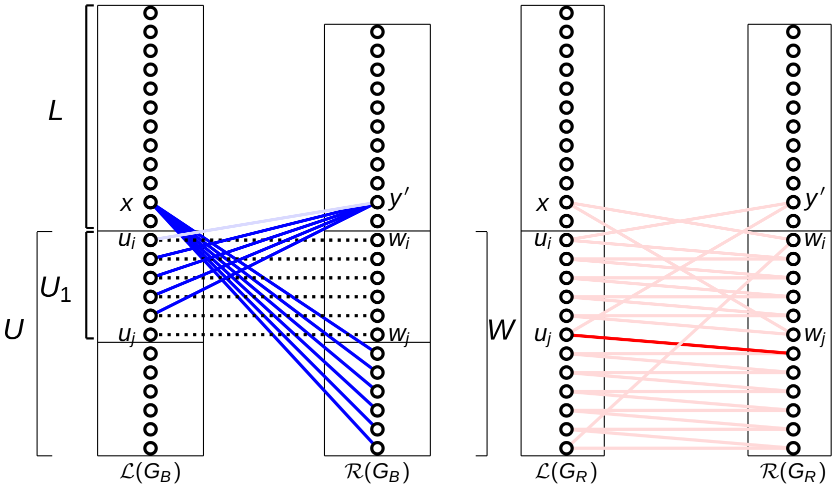
Let $x \in L$ ($y \in Y$, resp.) be a vertex that has exactly t blue neighbors in W (U, resp.), such that t is as small as possible. In G_B, there exists a set $U_1 \subseteq U$ ($W_1 \subseteq W$, resp.) such that $|U_1| = s - 1 - t$ ($|W_1| = s - 1 - t$, resp.) and, if $|U_1| \geq 1$ ($|W_1| \geq 1$, resp.), every vertex in Y (L, resp.) has $|U_1| - 1$ ($|W_1| - 1$, resp.) blue neighbors in U_1 (W_1, resp.), or a C_{2s} exists in G_R.

Lemma 5:

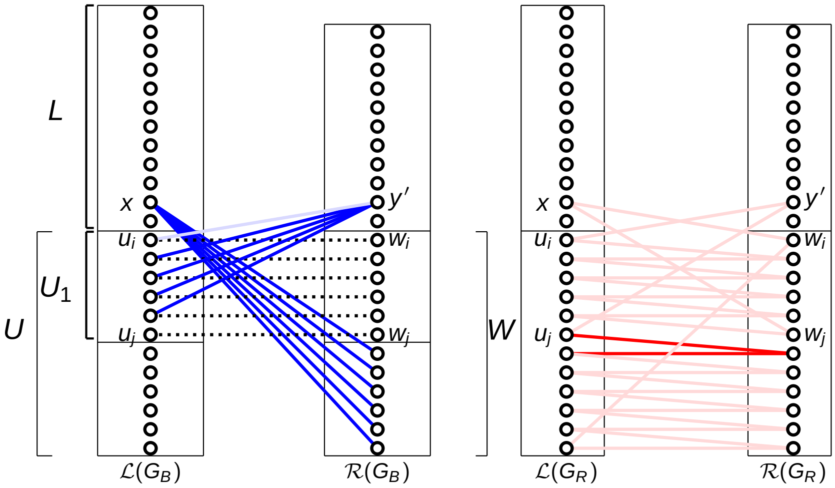
Let $x \in L$ ($y \in Y$, resp.) be a vertex that has exactly t blue neighbors in W (U, resp.), such that t is as small as possible. In G_B, there exists a set $U_1 \subseteq U$ ($W_1 \subseteq W$, resp.) such that $|U_1| = s - 1 - t$ ($|W_1| = s - 1 - t$, resp.) and, if $|U_1| \geq 1$ ($|W_1| \geq 1$, resp.), every vertex in Y (L, resp.) has $|U_1| - 1$ ($|W_1| - 1$, resp.) blue neighbors in U_1 (W_1, resp.), or a C_{2s} exists in G_R.



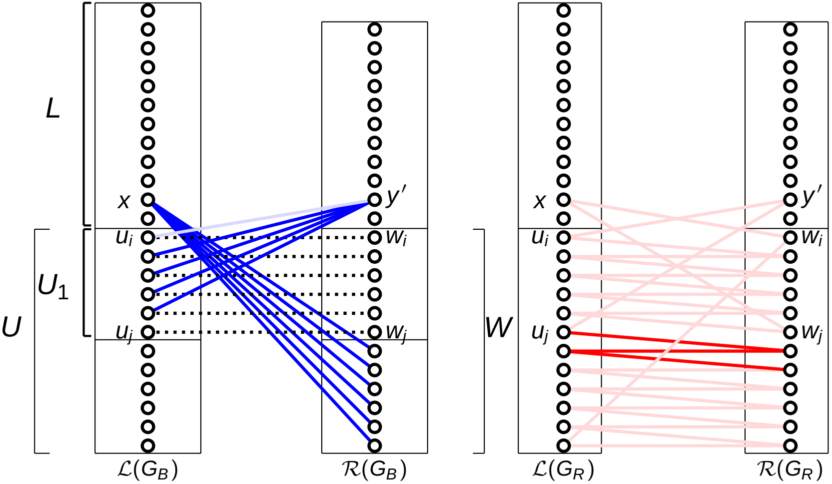
Lemma illustration.



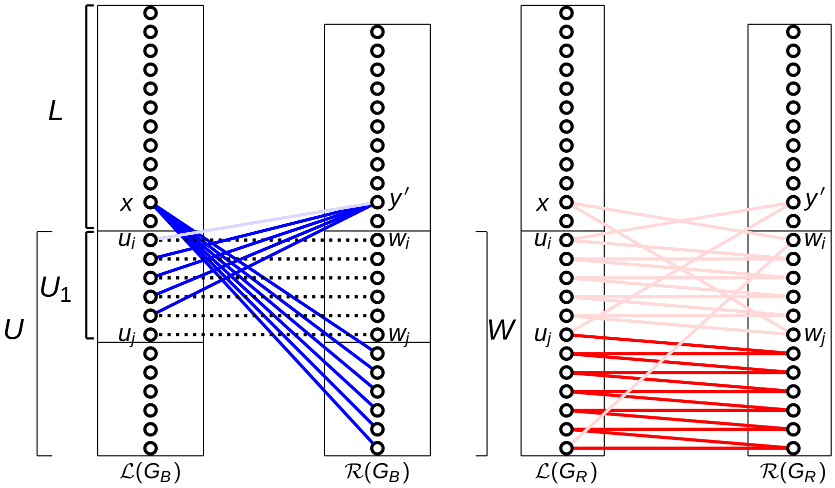
Lemma illustration.



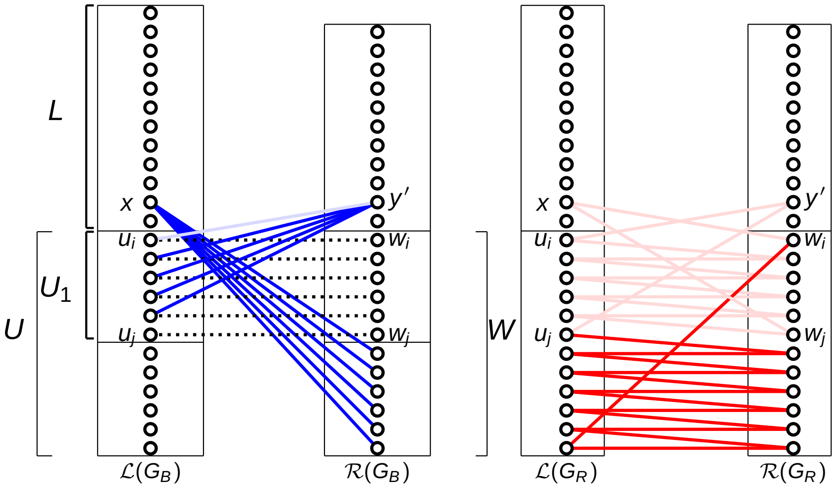
Lemma illustration.



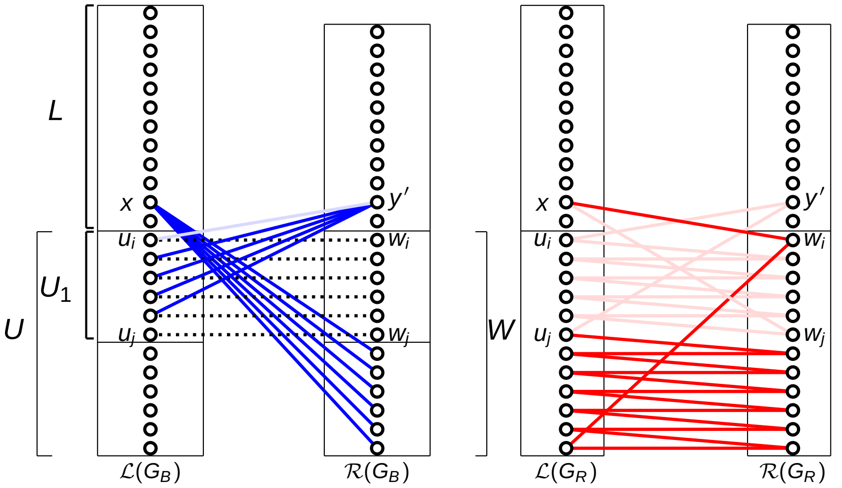
Lemma illustration.



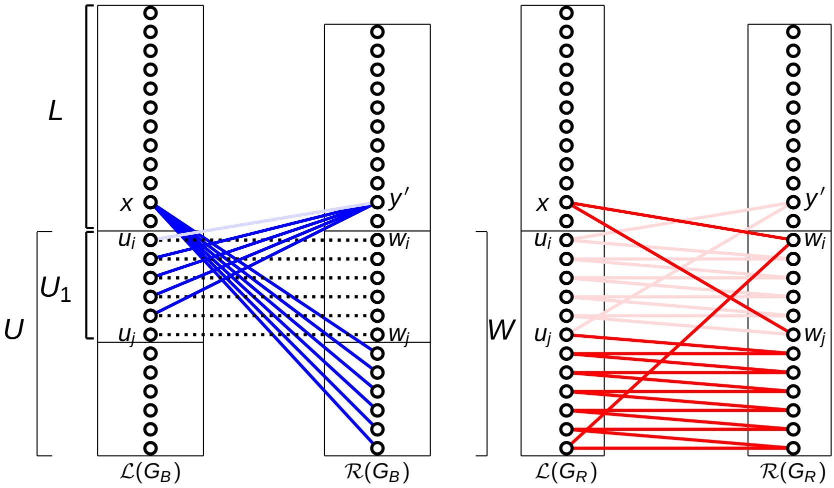
Lemma illustration.



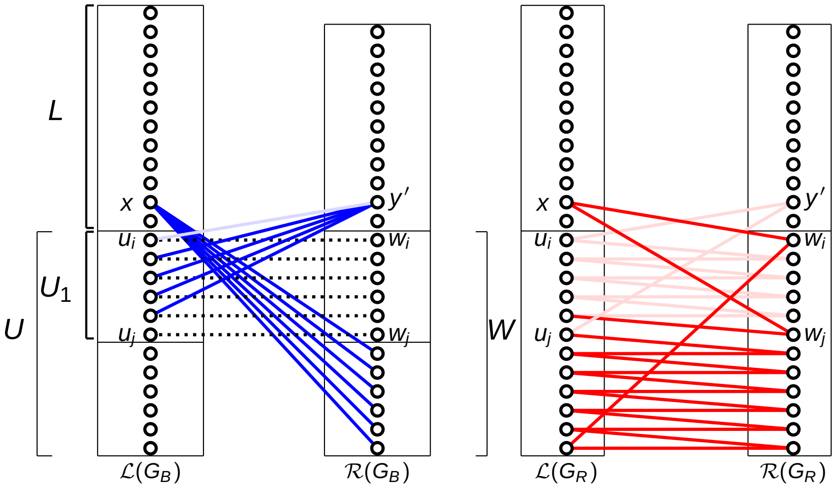
Lemma illustration.



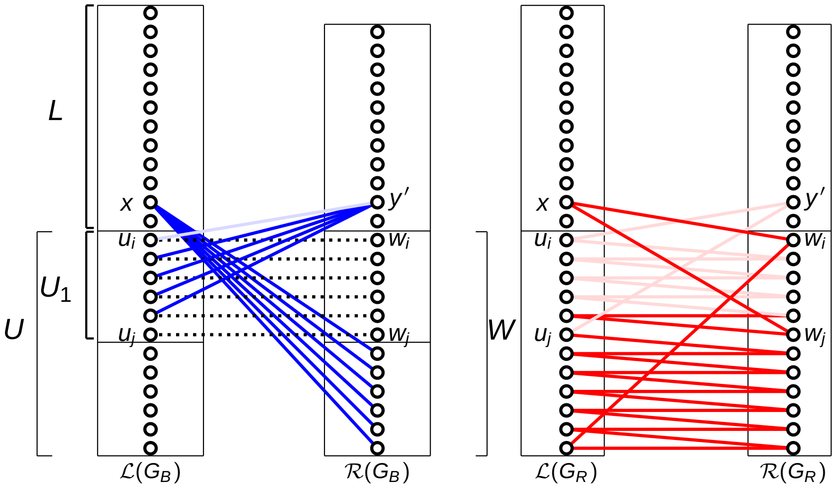
Lemma illustration.



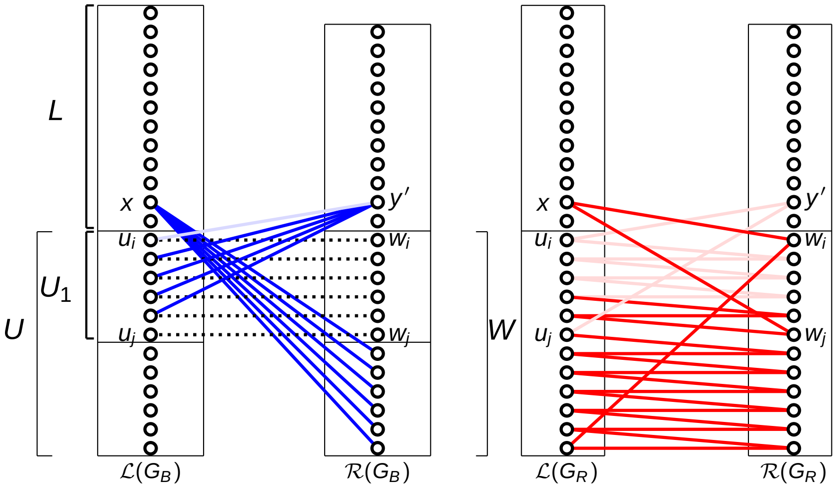
Lemma illustration.



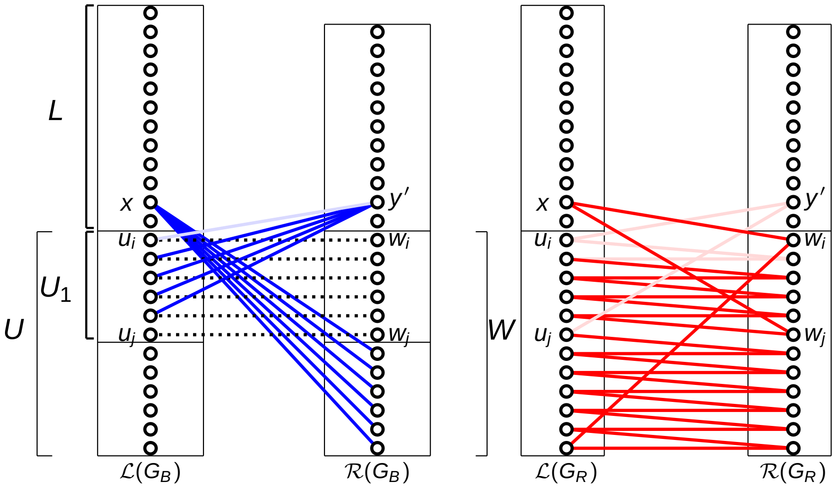
Lemma illustration.



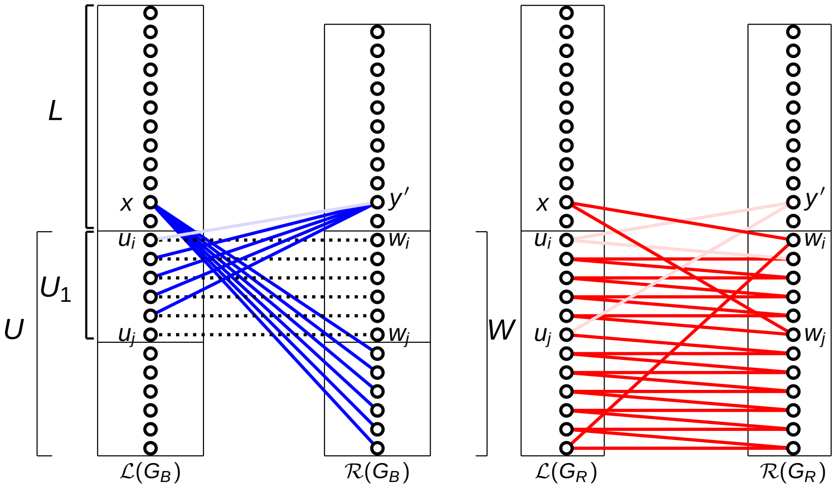
Lemma illustration.



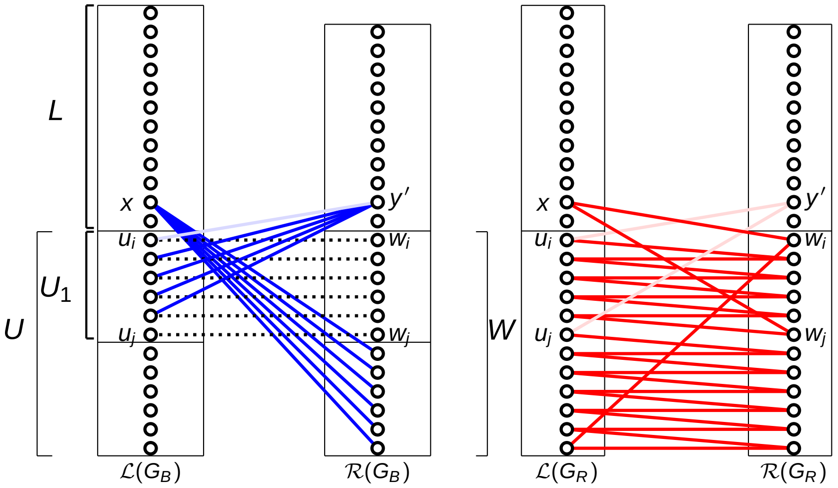
Lemma illustration.



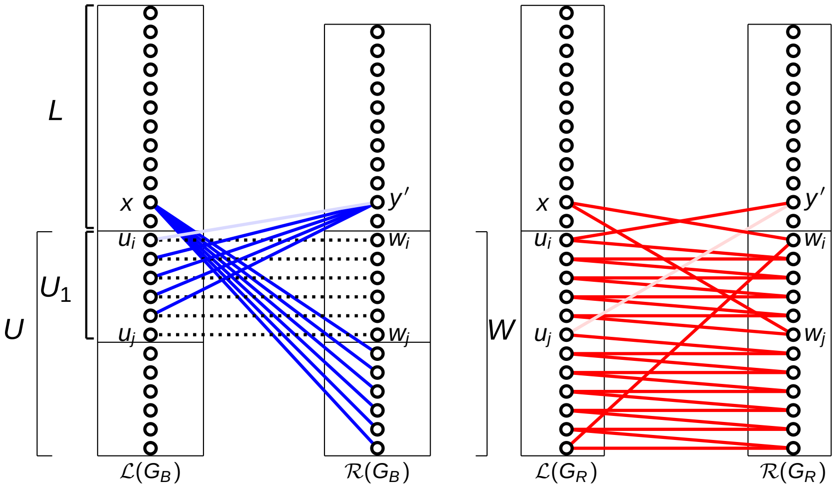
Lemma illustration.



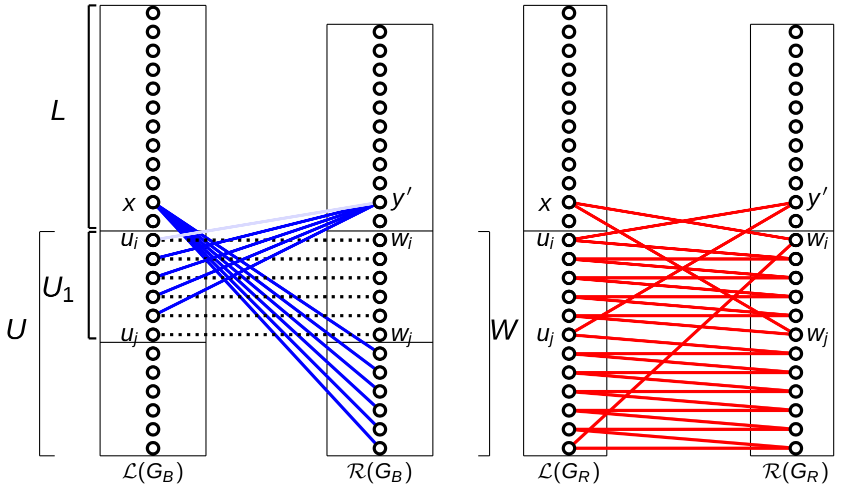
Lemma illustration.



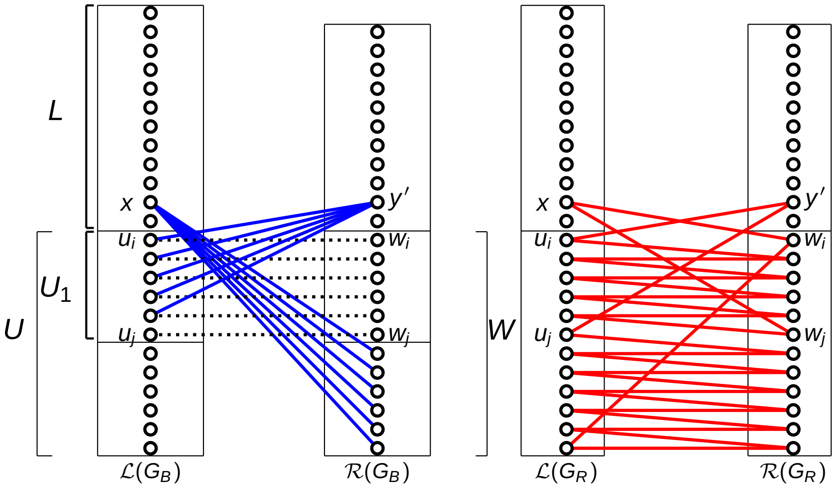
Lemma illustration.



Lemma illustration.



Lemma illustration.



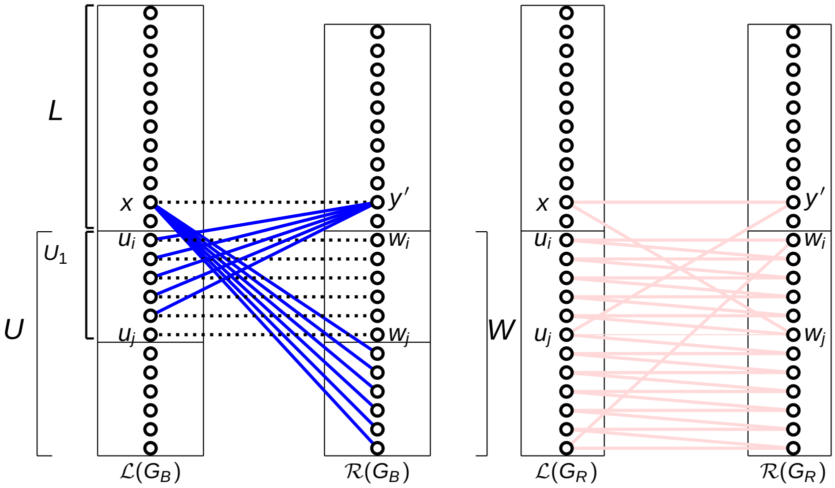
Lemma illustration.

Lemma 6:

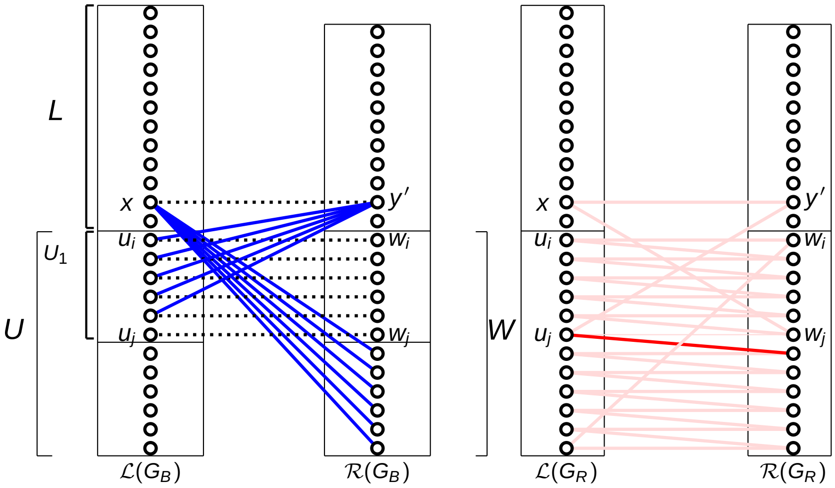
Let $x \in L$ ($y \in Y$, resp.) be a vertex that has exactly t blue neighbors in W (U, resp.), such that t is as small as possible. If $|U_1| \geq 1$ ($|W_1| \geq 1$, resp.), then, in G_B, if $y' \in Y$ ($x' \in L$, resp.) is a vertex not adjacent to x (y, resp.), then y' (x', resp.) must be adjacent to all vertices in U_1 (W_1, resp.), or a C_{2s} exists in G_R.

Lemma 6:

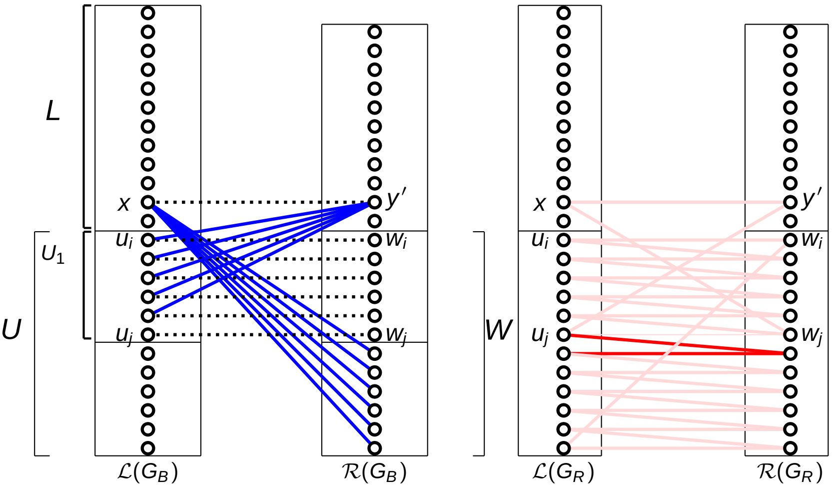
Let $x \in L$ ($y \in Y$, resp.) be a vertex that has exactly t blue neighbors in W (U, resp.), such that t is as small as possible. If $|U_1| \geq 1$ ($|W_1| \geq 1$, resp.), then, in G_B, if $y' \in Y$ ($x' \in L$, resp.) is a vertex not adjacent to x (y, resp.), then y' (x', resp.) must be adjacent to all vertices in U_1 (W_1, resp.), or a C_{2s} exists in G_R.



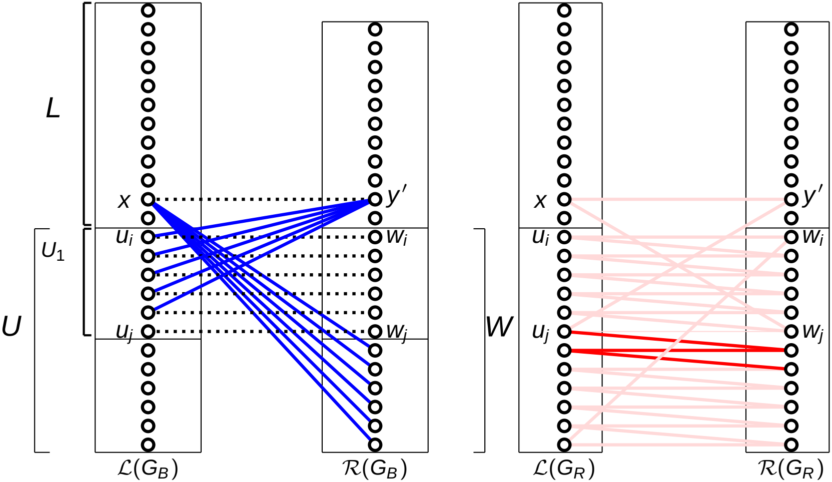
Lemma illustration.



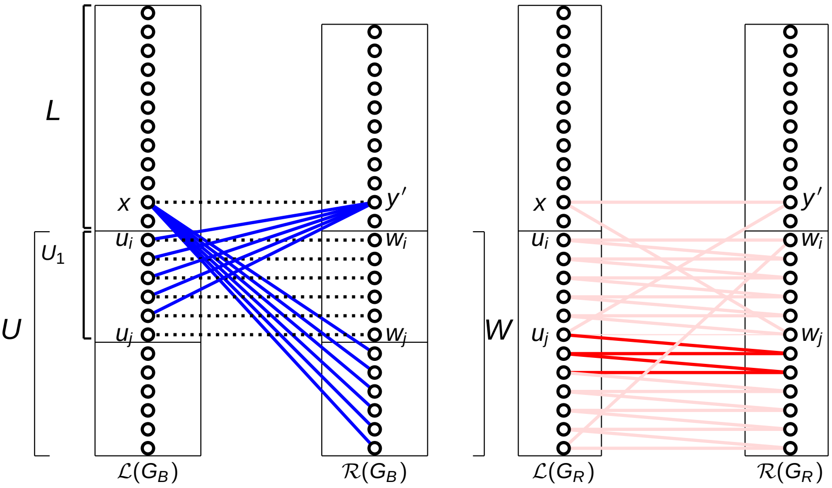
Lemma illustration.



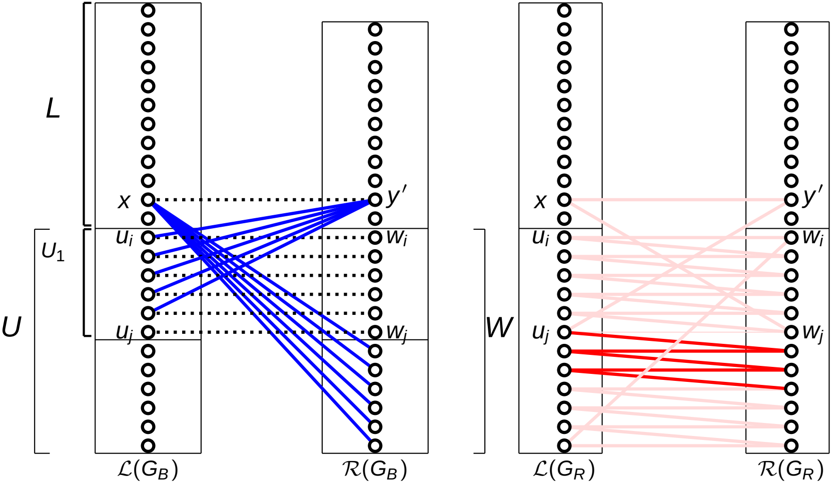
Lemma illustration.



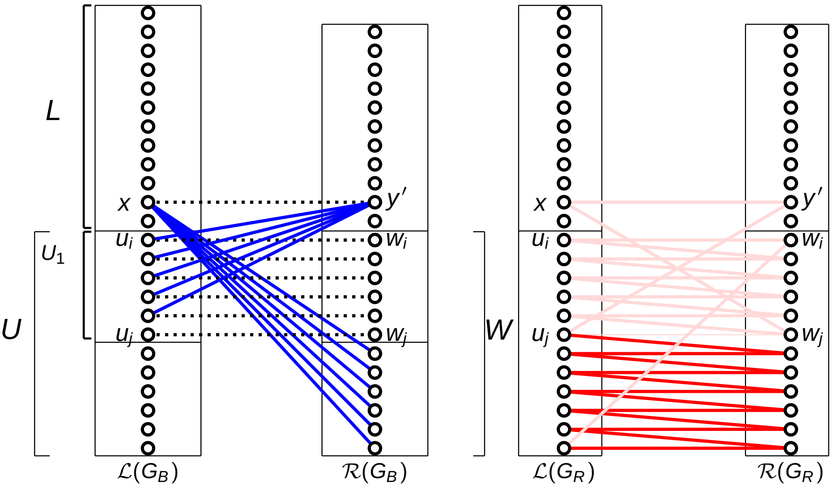
Lemma illustration.



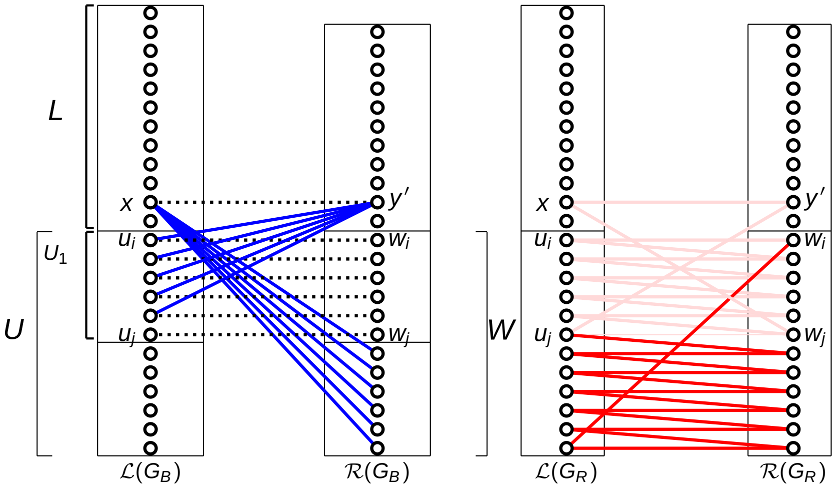
Lemma illustration.



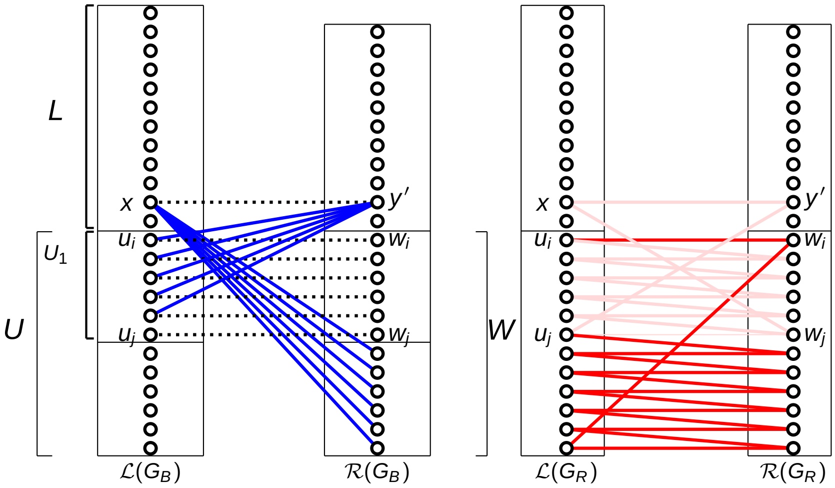
Lemma illustration.



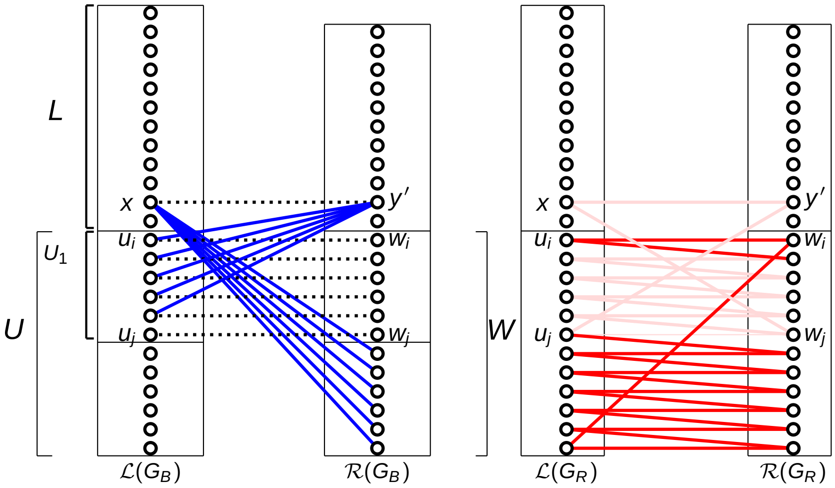
Lemma illustration.



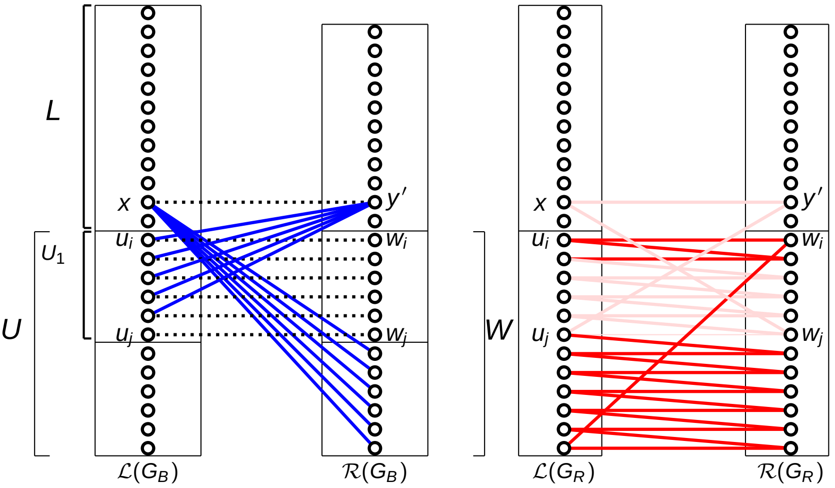
Lemma illustration.



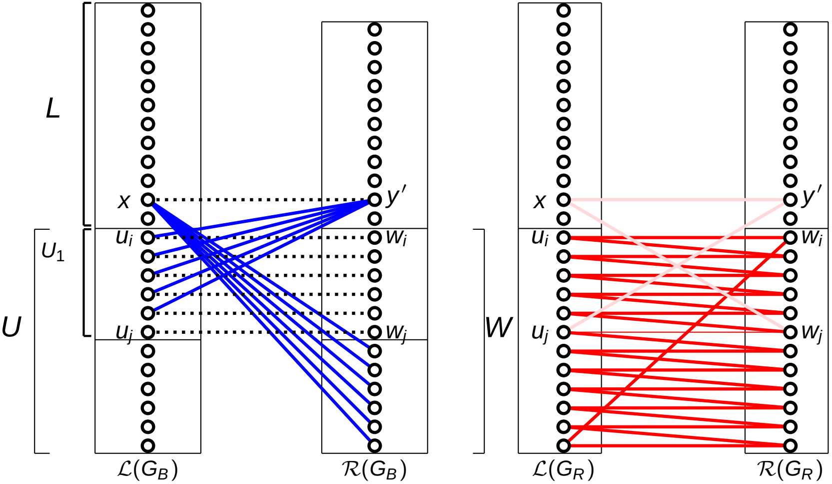
Lemma illustration.



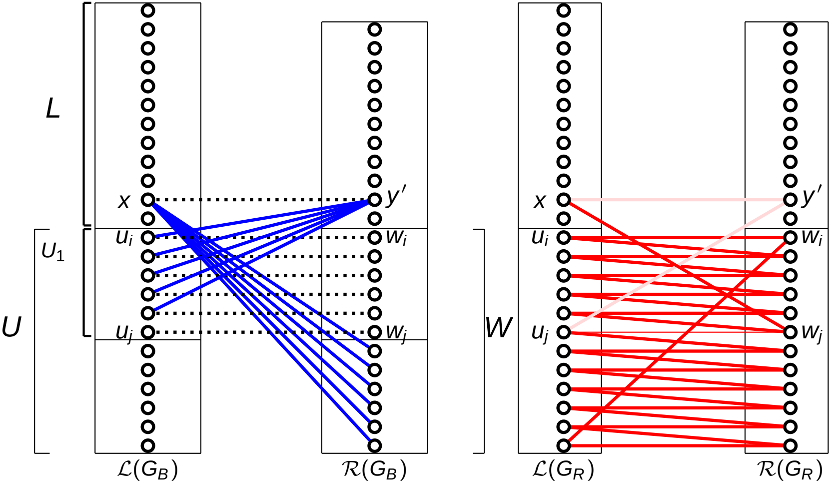
Lemma illustration.



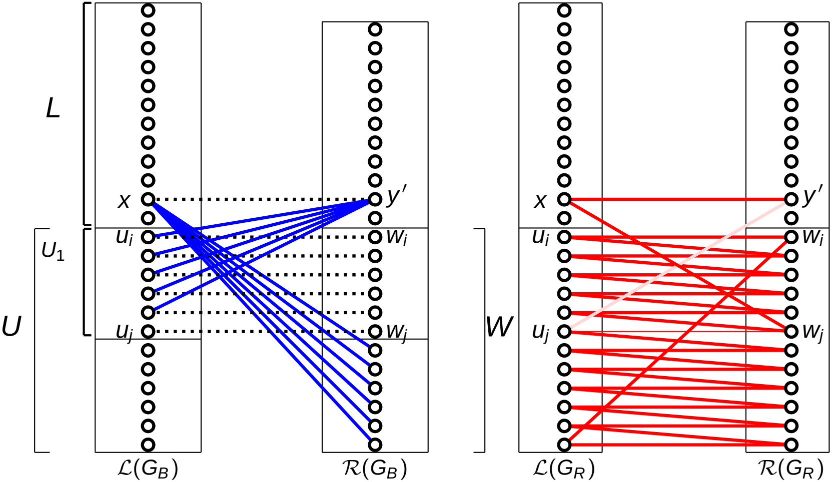
Lemma illustration.



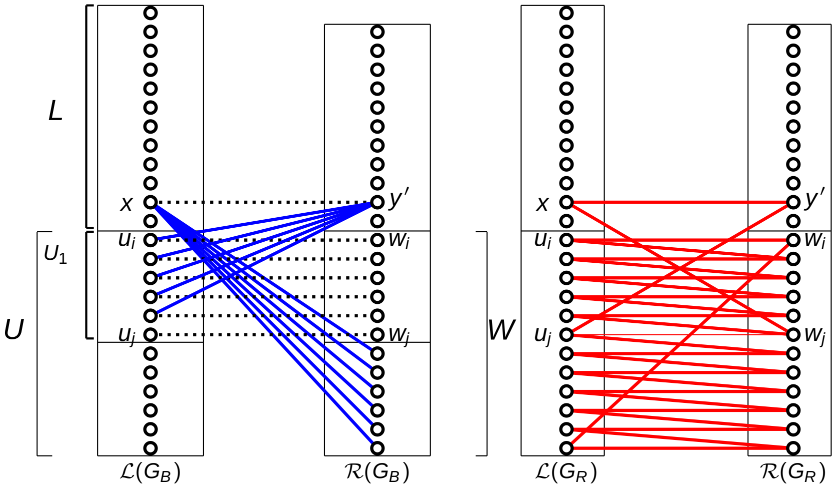
Lemma illustration.



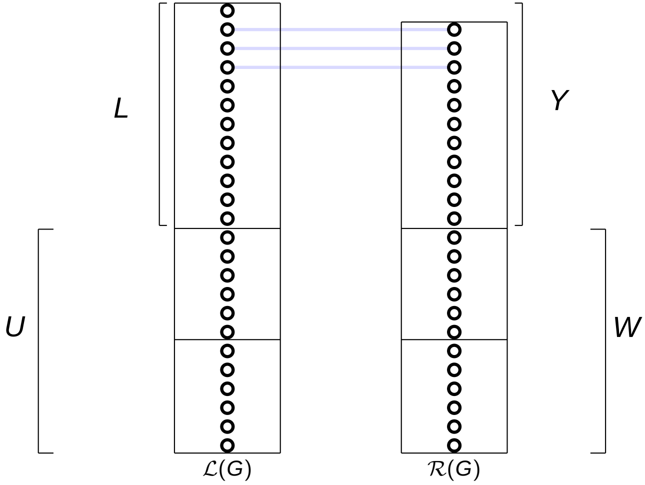
Lemma illustration.



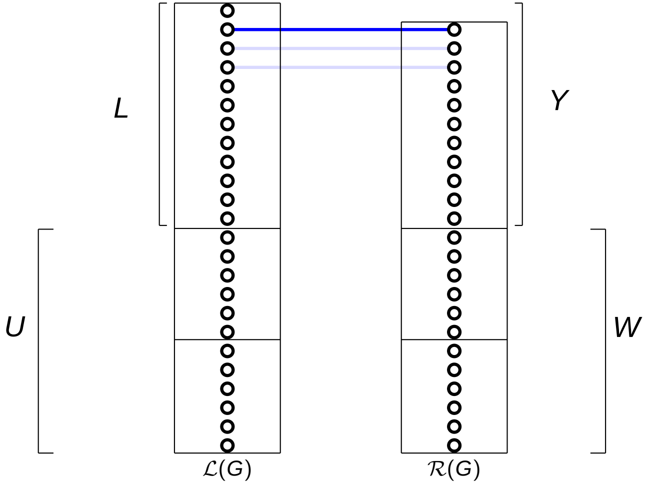
Lemma illustration.



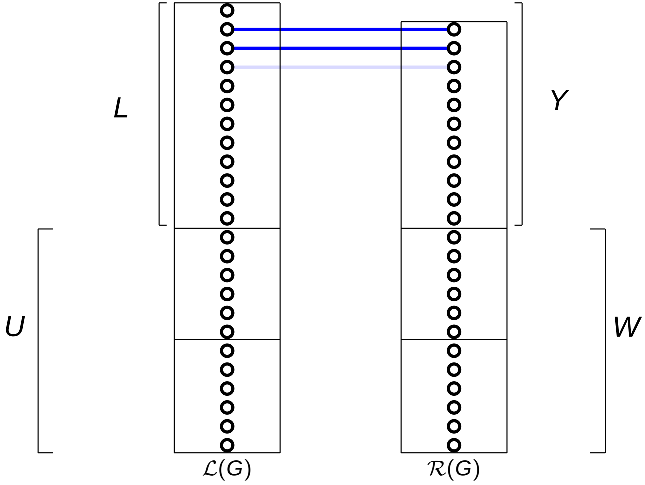
Lemma illustration.



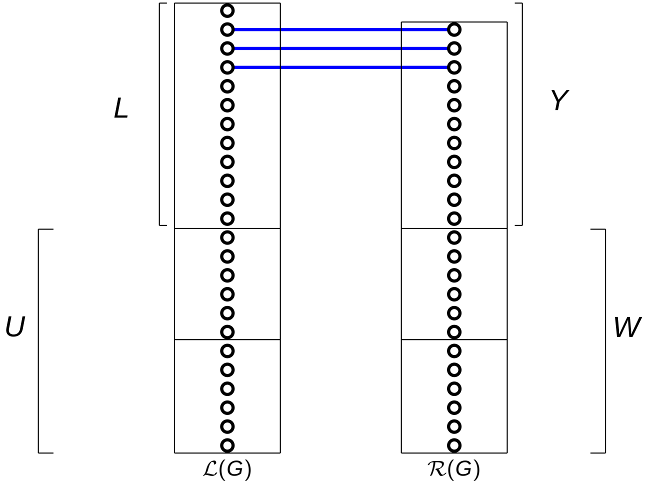
In addition, we can prove that $G_B \langle L \cup Y \rangle$ has three disjoint K_2's.



In addition, we can prove that $G_B \langle L \cup Y \rangle$ has three disjoint K_2's.



In addition, we can prove that $G_B \langle L \cup Y \rangle$ has three disjoint K_2's.



In addition, we can prove that $G_B \langle L \cup Y \rangle$ has three disjoint K_2's.

Proof illustration using a single case

- Consider a blue-red-coloring of $G = K_{2s,2s-1}$ such that G_R has a C_{2s-2}. Define the sets $U = \mathcal{L}(G) \cap V(C)$, $W = \mathcal{R}(G) \cap V(C)$, $L = \mathcal{L}(G) - U$ and $Y = \mathcal{R}(G) - W$.
- Pick $x \in L$ ($y \in Y$, resp.) with t (t', resp.) blue neighbors in W (U, resp.), such that t (t', resp.) is as small as possible.
- Sets U_1 and W_1 exist such that every vertex in Y (L, resp.) is adjacent to $|U_1| - 1$ ($|W_1| - 1$, resp.) vertices in U_1 (W_1, resp.).
- Note that $|U_1| = s - 1 - t$ and $|W_1| = s - 1 - t'$. Define $W_2 = W - W_1$ and $U_2 = U - U_1$.

Proof illustration using a single case

- Consider a blue-red-coloring of $G = K_{2s,2s-1}$ such that G_R has a C_{2s-2}. Define the sets $U = \mathcal{L}(G) \cap V(C)$, $W = \mathcal{R}(G) \cap V(C)$, $L = \mathcal{L}(G) - U$ and $Y = \mathcal{R}(G) - W$.
- Pick $x \in L$ ($y \in Y$, resp.) with t (t', resp.) blue neighbors in W (U, resp.), such that t (t', resp.) is as small as possible.
- Sets U_1 and W_1 exist such that every vertex in Y (L, resp.) is adjacent to $|U_1| - 1$ ($|W_1| - 1$, resp.) vertices in U_1 (W_1, resp.).
- Note that $|U_1| = s - 1 - t$ and $|W_1| = s - 1 - t'$. Define $W_2 = W - W_1$ and $U_2 = U - U_1$.

Proof illustration using a single case

- Consider a blue-red-coloring of $G = K_{2s,2s-1}$ such that G_R has a C_{2s-2}. Define the sets $U = \mathcal{L}(G) \cap V(C)$, $W = \mathcal{R}(G) \cap V(C)$, $L = \mathcal{L}(G) - U$ and $Y = \mathcal{R}(G) - W$.
- Pick $x \in L$ ($y \in Y$, resp.) with t (t', resp.) blue neighbors in W (U, resp.), such that t (t', resp.) is as small as possible.
- Sets U_1 and W_1 exist such that every vertex in Y (L, resp.) is adjacent to $|U_1| - 1$ ($|W_1| - 1$, resp.) vertices in U_1 (W_1, resp.).
- Note that $|U_1| = s - 1 - t$ and $|W_1| = s - 1 - t'$. Define $W_2 = W - W_1$ and $U_2 = U - U_1$.

Proof illustration using a single case

- Consider a blue-red-coloring of $G = K_{2s,2s-1}$ such that G_R has a C_{2s-2}. Define the sets $U = \mathcal{L}(G) \cap V(C)$, $W = \mathcal{R}(G) \cap V(C)$, $L = \mathcal{L}(G) - U$ and $Y = \mathcal{R}(G) - W$.
- Pick $x \in L$ ($y \in Y$, resp.) with t (t', resp.) blue neighbors in W (U, resp.), such that t (t', resp.) is as small as possible.
- Sets U_1 and W_1 exist such that every vertex in Y (L, resp.) is adjacent to $|U_1| - 1$ ($|W_1| - 1$, resp.) vertices in U_1 (W_1, resp.).
- Note that $|U_1| = s - 1 - t$ and $|W_1| = s - 1 - t'$. Define $W_2 = W - W_1$ and $U_2 = U - U_1$.

- For convenience, let X be a subset of L with s vertices such that X contains x and $G_B \langle X \cup Y \rangle$ contains the three disjoint K_2's.
- For illustration purposes, we will consider only a tiny part of one of the three main cases. We will also only consider s to be even:
- ***Case 1.*** *There exists integers $i, j \geq 1$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$:*

  $|U_1| = s - 1 - t = s/2 - 1 - i$, $|U_2| = t = s/2 + i$,
  $|W_1| = s - 1 - t' = s/2 - 1 - j$ and $|W_2| = t' = s/2 + j$

- For convenience, let X be a subset of L with s vertices such that X contains x and $G_B\langle X \cup Y\rangle$ contains the three disjoint K_2's.

- For illustration purposes, we will consider only a tiny part of one of the three main cases. We will also only consider s to be even:

- ***Case 1.*** *There exists integers $i, j \geq 1$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$:*

  $|U_1| = s - 1 - t = s/2 - 1 - i$, $|U_2| = t = s/2 + i$, $|W_1| = s - 1 - t' = s/2 - 1 - j$ and $|W_2| = t' = s/2 + j$

- For convenience, let X be a subset of L with s vertices such that X contains x and $G_B \langle X \cup Y \rangle$ contains the three disjoint K_2's.
- For illustration purposes, we will consider only a tiny part of one of the three main cases. We will also only consider s to be even:
- ***Case 1.*** *There exists integers $i, j \geq 1$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$:*

  $|U_1| = s - 1 - t = s/2 - 1 - i$, $|U_2| = t = s/2 + i$,
  $|W_1| = s - 1 - t' = s/2 - 1 - j$ and $|W_2| = t' = s/2 + j$

- For convenience, let X be a subset of L with s vertices such that X contains x and $G_B\langle X \cup Y \rangle$ contains the three disjoint K_2's.
- For illustration purposes, we will consider only a tiny part of one of the three main cases. We will also only consider s to be even:
- ***Case 1.*** *There exists integers $i, j \geq 1$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$:*

$|U_1| = s - 1 - t = s/2 - 1 - i$, $|U_2| = t = s/2 + i$,
$|W_1| = s - 1 - t' = s/2 - 1 - j$ and $|W_2| = t' = s/2 + j$

- For convenience, let X be a subset of L with s vertices such that X contains x and $G_B\langle X \cup Y\rangle$ contains the three disjoint K_2's.
- For illustration purposes, we will consider only a tiny part of one of the three main cases. We will also only consider s to be even:
- ***Case 1.*** *There exists integers $i, j \geq 1$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$:*

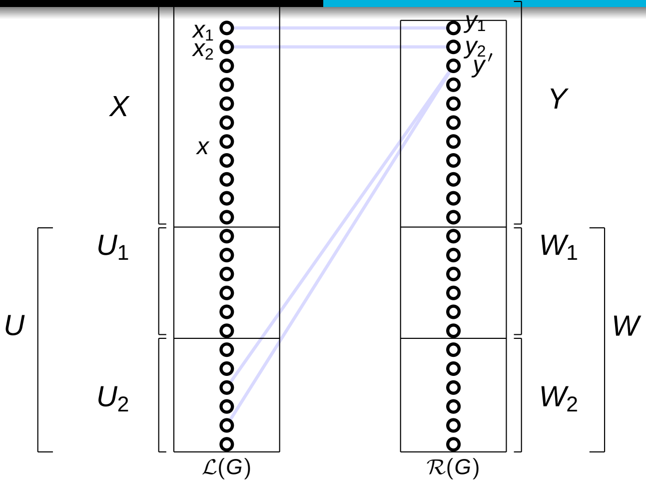
  $|U_1| = s - 1 - t = s/2 - 1 - i$, $|U_2| = t = s/2 + i$,
  $|W_1| = s - 1 - t' = s/2 - 1 - j$ and $|W_2| = t' = s/2 + j$

- ***Case 1.1.*** $s/2 - j - 2 > j + 2$**:**
- ***Case 1.1.2.*** $s/2 - i - 2 \leq i + 2$**:**
- $s/4 - 2 \leq i$

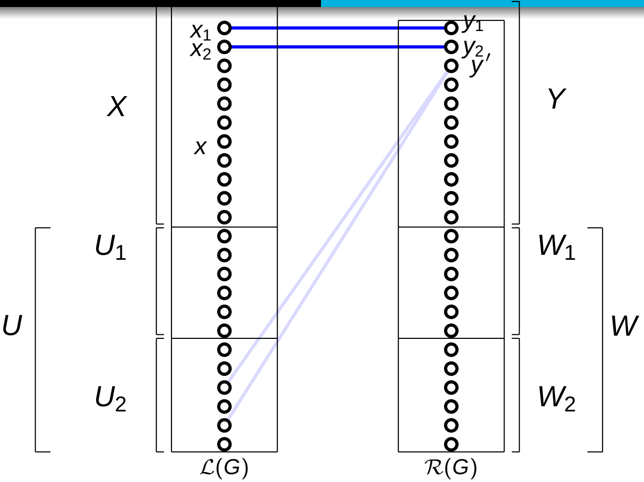
- ***Case 1.1.*** $s/2 - j - 2 > j + 2$***:***
- ***Case 1.1.2.*** $s/2 - i - 2 \leq i + 2$***:***
- $s/4 - 2 \leq i$

- ***Case 1.1.*** $s/2 - j - 2 > j + 2$***:***
- ***Case 1.1.2.*** $s/2 - i - 2 \leq i + 2$***:***
- $s/4 - 2 \leq i$

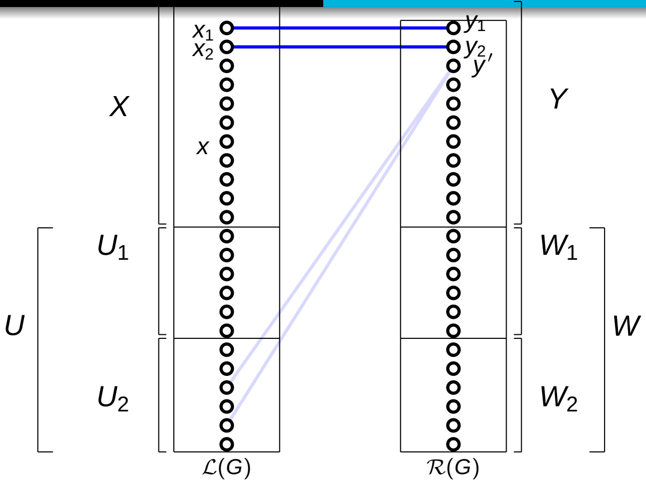
- ***Case 1.1.*** $s/2 - j - 2 > j + 2$***:***
- ***Case 1.1.2.*** $s/2 - i - 2 \leq i + 2$***:***
- $s/4 - 2 \leq i$



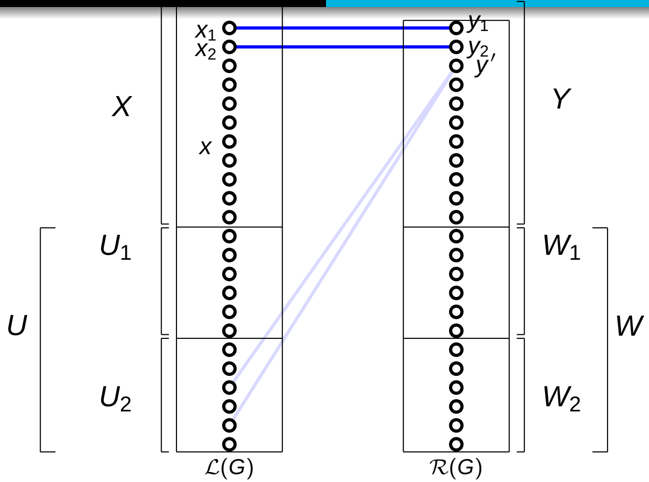
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$ $\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



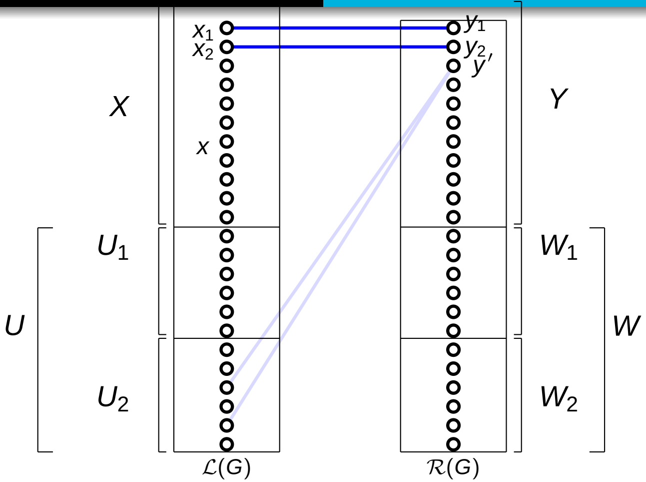
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$
$\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



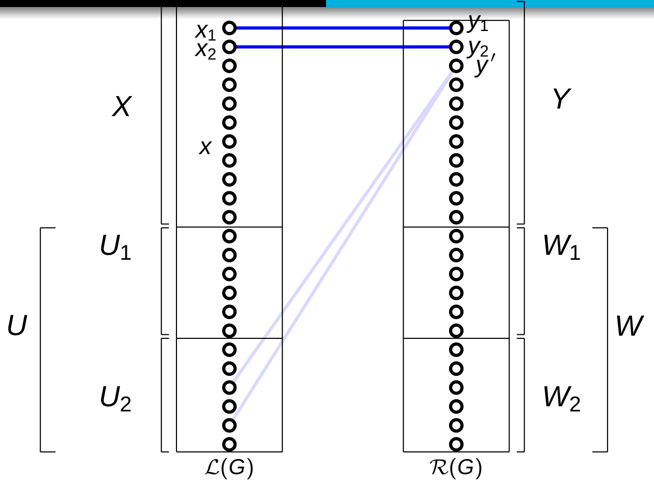
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1$ $\Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$ $\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



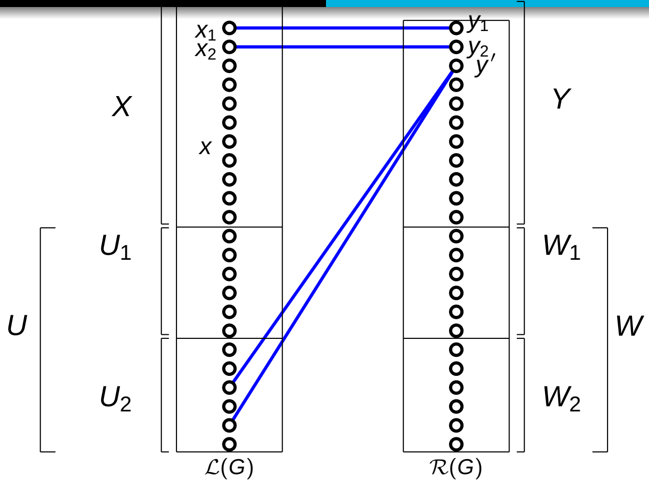
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$ $\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



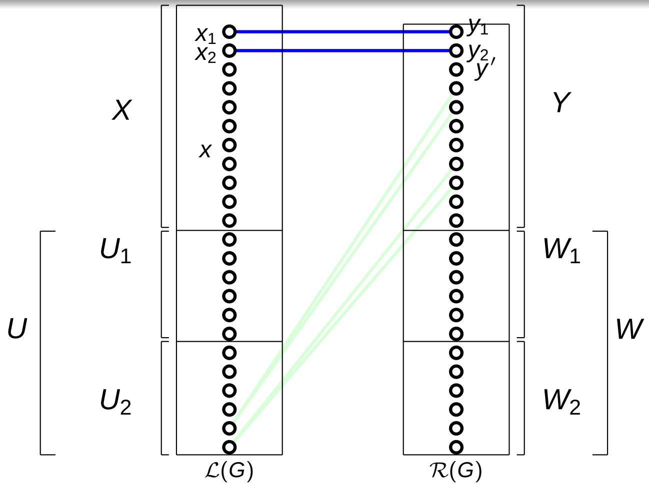
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$
$\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



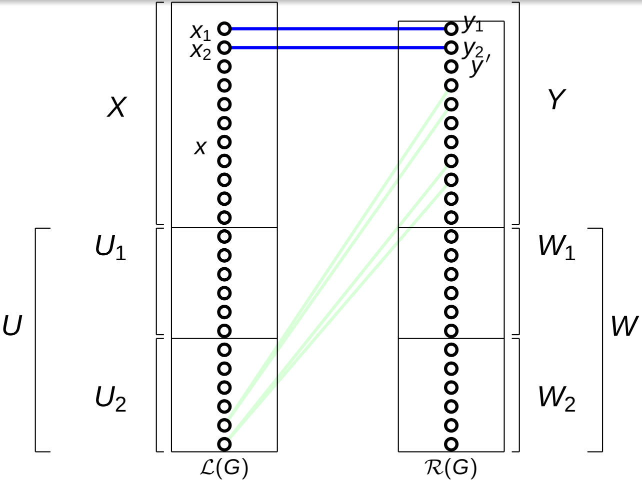
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$
$\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



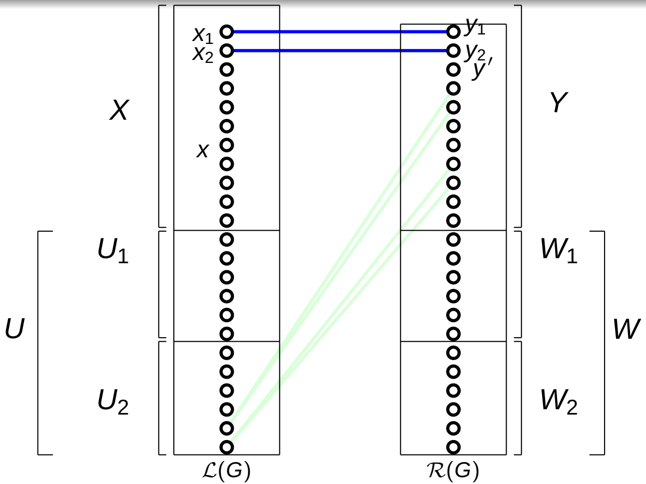
Set $S'' = Y - \{y_1, y_2\}$, $T'' = U_2$ and $k'' = 1 \Longrightarrow$ As $i \geq s/4 - 2$ and $s \geq 18$, $y' \in Y \Longrightarrow$
$\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j \geq 2 \Longrightarrow y'$ has more than one neighbor in T''



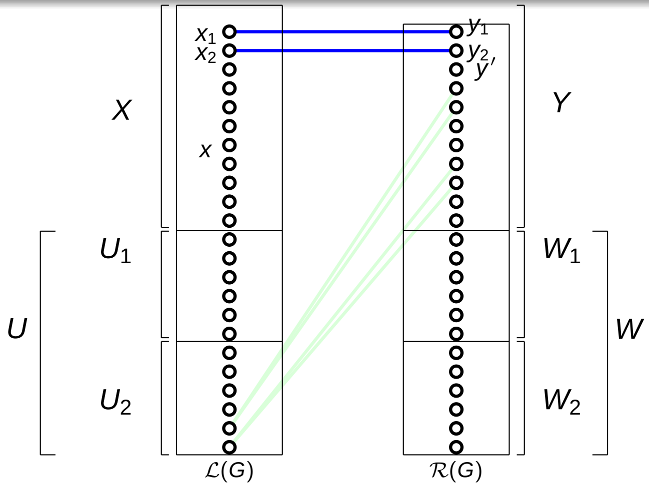
$|U_1| = s - 1 - t \geq 3 \implies |S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''| \implies$ there exists $k'' + 1 = 2$ copies of $K_{1,2}$ with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$



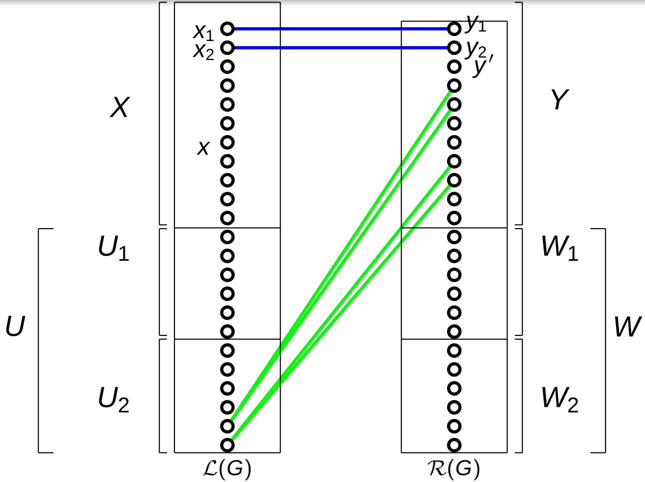
$|U_1| = s - 1 - t \geq 3 \Longrightarrow |S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''| \Longrightarrow$ there exists $k'' + 1 = 2$ copies of $K_{1,2}$ with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$



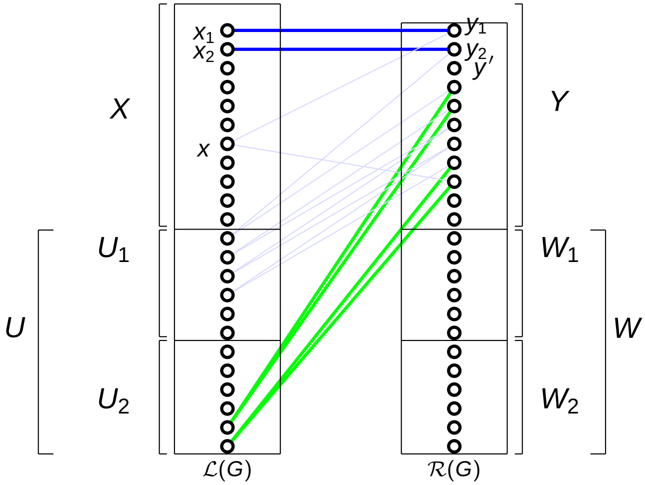
$|U_1| = s - 1 - t \geq 3 \Longrightarrow |S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''| \Longrightarrow$ there exists $k'' + 1 = 2$ copies of $K_{1,2}$ with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$



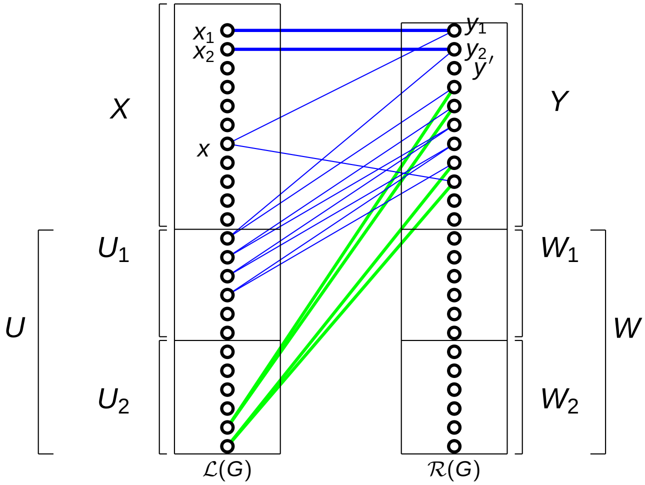
$|U_1| = s - 1 - t \geq 3 \Longrightarrow |S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''| \Longrightarrow$ there exists $k'' + 1 = 2$ copies of $K_{1,2}$ with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$



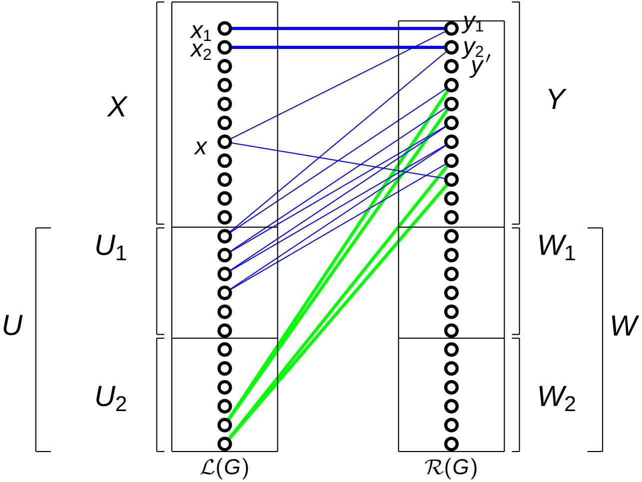
$|U_1| = s - 1 - t \geq 3 \Longrightarrow |S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''| \Longrightarrow$ there exists $k'' + 1 = 2$ copies of $K_{1,2}$ with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$



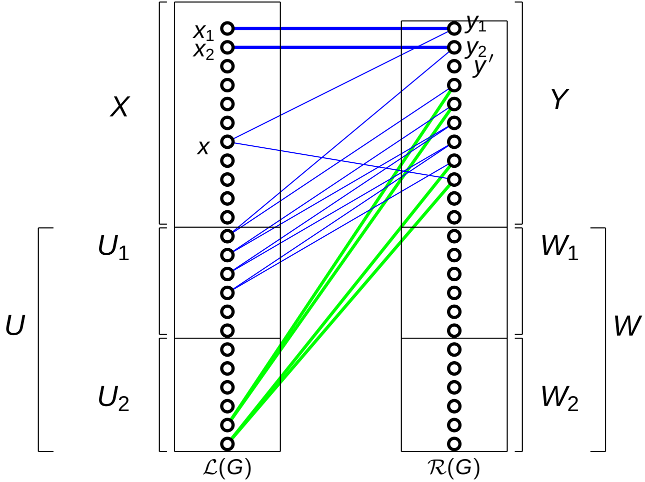
The z construction is satisfied so there exists a $y_2 - y_1$ path P' that has $2|U_1| - 1 + k + \ell = 2(s/2 - i - 1) - 1 + 3 + 3 = s - 2i + 3$ vertices



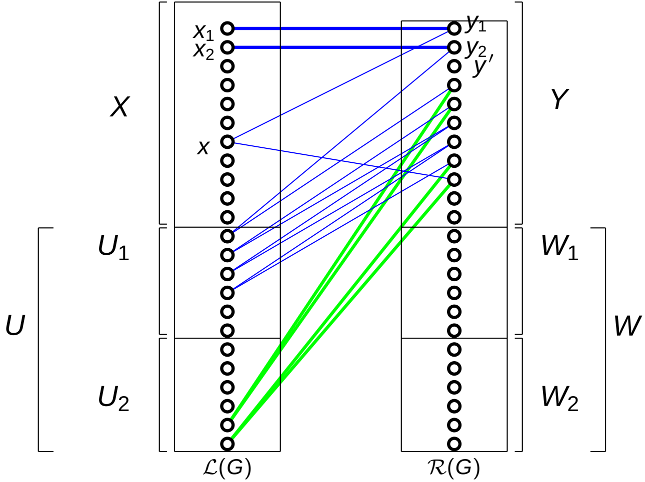
The z construction is satisfied so there exists a $y_2 - y_1$ path P' that has $2|U_1| - 1 + k + \ell = 2(s/2 - i - 1) - 1 + 3 + 3 = s - 2i + 3$ vertices



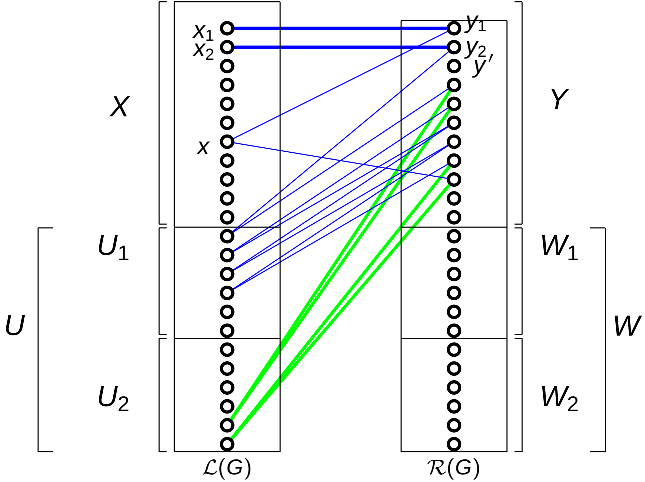
For every $x' \in X - \{x, x_1, x_2\} \Longrightarrow$
$\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j \Longrightarrow$ For
$G' = G_B \langle X - \{x, x_1, x_2\} \rangle$ we have $m(G') \geq (s-3)(i+j)$



For every $x' \in X - \{x, x_1, x_2\} \implies$
$\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j \implies$ For
$G' = G_B \langle X - \{x, x_1, x_2\} \rangle$ we have $m(G') \geq (s-3)(i+j)$



For every $x' \in X - \{x, x_1, x_2\} \Longrightarrow$
$\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j \Longrightarrow$ For
$G' = G_B \langle X - \{x, x_1, x_2\} \rangle$ we have $m(G') \geq (s-3)(i+j)$



For every $x' \in X - \{x, x_1, x_2\} \implies$
$\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j \implies$ For
$G' = G_B \langle X - \{x, x_1, x_2\} \rangle$ we have $m(G') \geq (s-3)(i+j)$

Theorem 2:

Let c be a positive non-zero integer and $G(S,T)$ be a bipartite graph with $|T| = a$ and $|S| = b$ ($a \leq b$). If G contains no path $P_{2\ell}$ for $\ell > c$, then

$$m(G(S,T)) \leq \begin{cases} ab, & \text{if } a \leq c, \\ bc, & \text{if } c < a < 2c, \\ (a+b-2c)c, & \text{if } a \geq 2c. \end{cases}$$

Theorem 2:

Let c be a positive non-zero integer and $G(S,T)$ be a bipartite graph with $|T| = a$ and $|S| = b$ ($a \leq b$). If G contains no path $P_{2\ell}$ for $\ell > c$, then

$$m(G(S,T)) \leq \begin{cases} ab, & \text{if } a \leq c, \\ bc, & \text{if } c < a < 2c, \\ (a+b-2c)c, & \text{if } a \geq 2c. \end{cases}$$

Theorem 2:

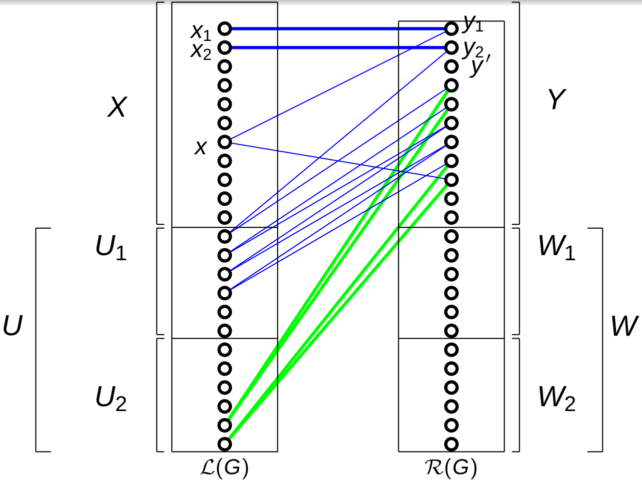
Let c be a positive non-zero integer and $G(S,T)$ be a bipartite graph with $|T| = a$ and $|S| = b$ ($a \leq b$). If G contains no path $P_{2\ell}$ for $\ell > c$, then

$$m(G(S,T)) \leq \begin{cases} ab, & \text{if } a \leq c, \\ bc, & \text{if } c < a < 2c, \\ (a+b-2c)c, & \text{if } a \geq 2c. \end{cases}$$

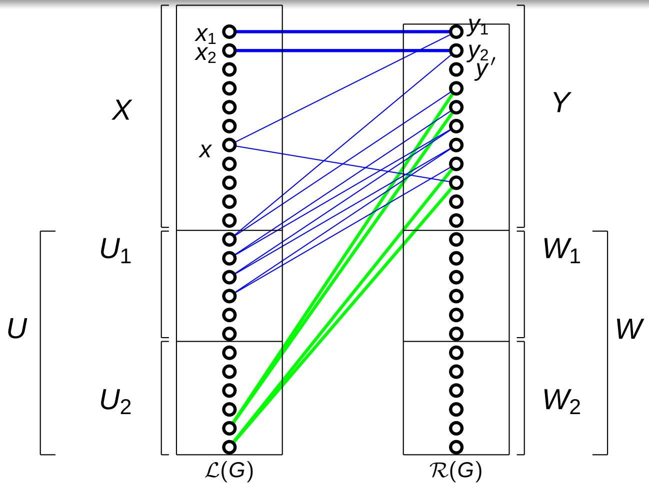
Theorem 2:

Let c be a positive non-zero integer and $G(S,T)$ be a bipartite graph with $|T| = a$ and $|S| = b$ ($a \leq b$). If G contains no path $P_{2\ell}$ for $\ell > c$, then

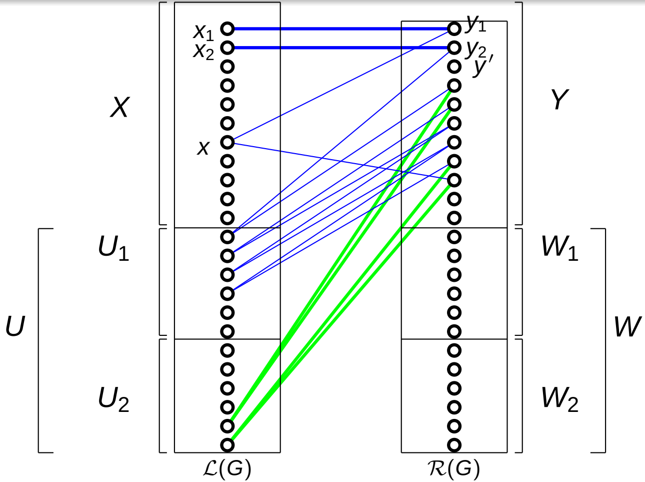
$$m(G(S,T)) \leq \begin{cases} ab, & \text{if } a \leq c, \\ bc, & \text{if } c < a < 2c, \\ (a+b-2c)c, & \text{if } a \geq 2c. \end{cases}$$



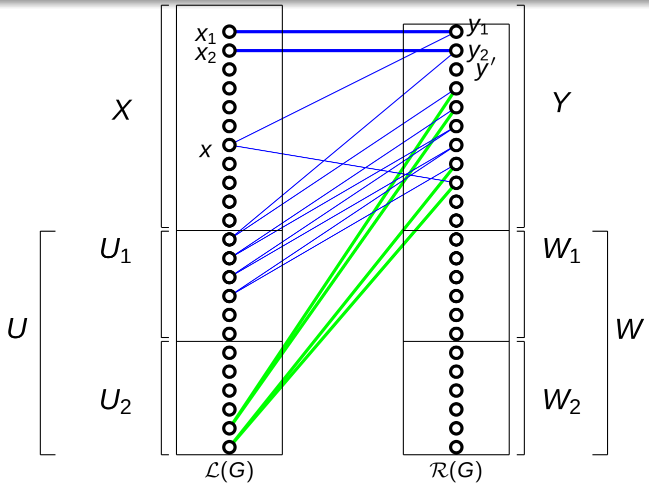
For $(a+b-2c)c \Longrightarrow$ define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \Longrightarrow$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \Longrightarrow$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\Longrightarrow$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction.



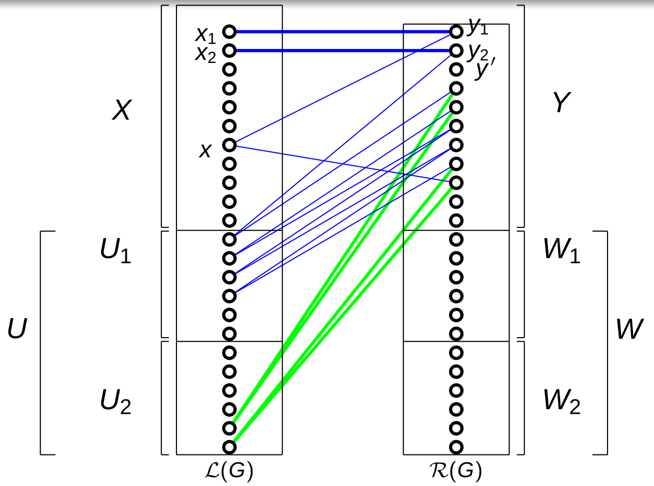
For $(a+b-2c)c \Longrightarrow$ define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \Longrightarrow$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \Longrightarrow$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\Longrightarrow$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction.



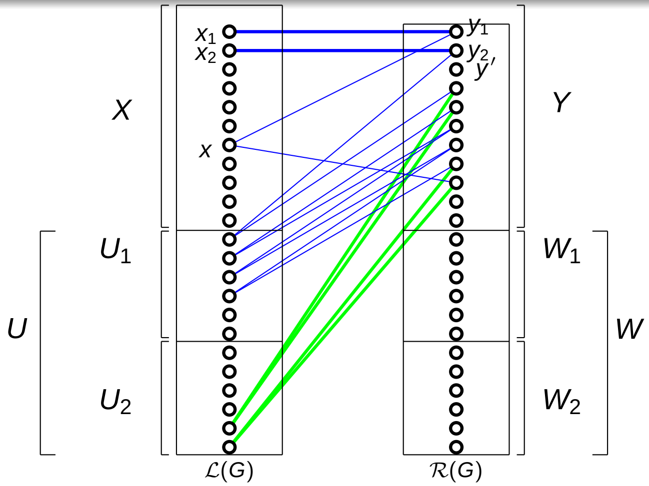
For $(a+b-2c)c \Longrightarrow$ define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \Longrightarrow$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \Longrightarrow$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\Longrightarrow$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction



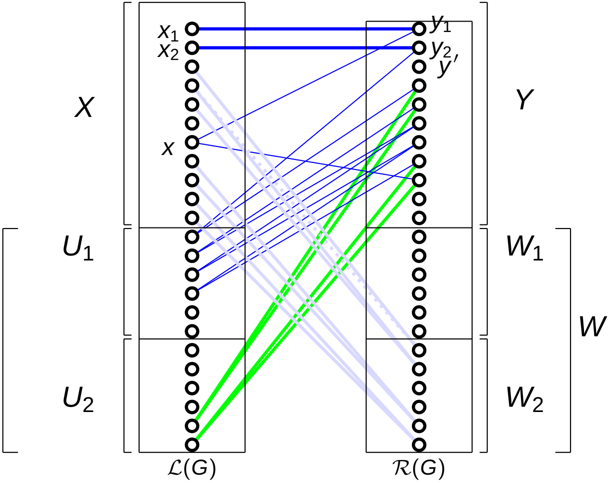
For $(a+b-2c)c \Longrightarrow$ define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \Longrightarrow$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \Longrightarrow$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\Longrightarrow$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction.



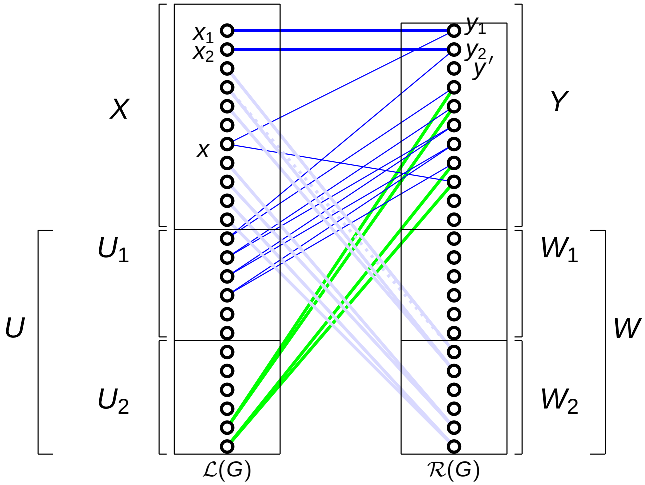
For $(a+b-2c)c \Longrightarrow$define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \Longrightarrow$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \Longrightarrow$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\Longrightarrow$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction.



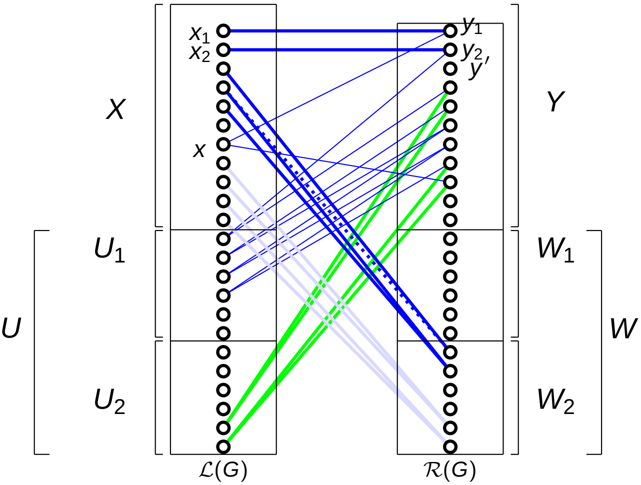
For $(a+b-2c)c \implies$ define $S = X - \{x, x_1, x_2\}$ and $T = W_2 \implies$ Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j \implies$ Pick $c = i + j - 2$ and assume G' has no path on $2(i+j-1)$ vertices $\implies$ Using calculus, we show $m(G') \geq (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, a contradiction



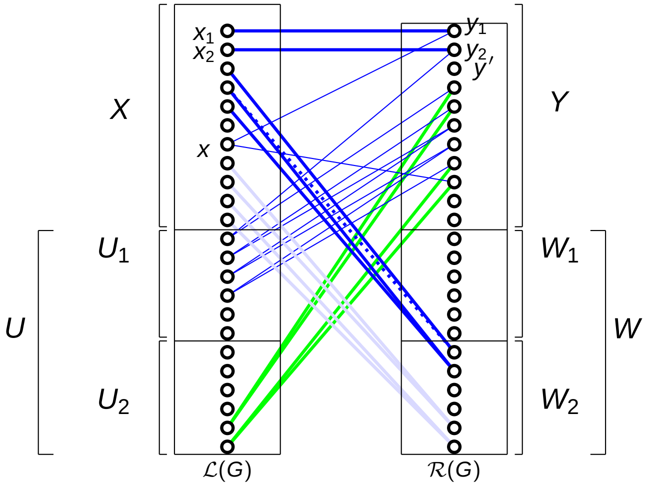
The graph G' has a path on $2(i+j-1)$ vertices and so has a path $P(3)$, that starts and ends in $X - \{x, x_1, x_2\}$, on $2(i+j) - 3$ vertices $\Longrightarrow$ We can also show again that $G' - V(P(3))$ has two disjoint $K_{1,2}$'s



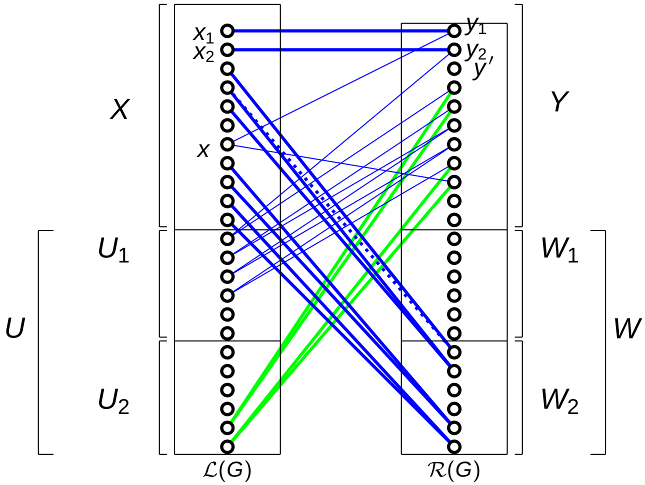
The graph G' has a path on $2(i+j-1)$ vertices and so has a path $P(3)$, that starts and ends in $X-\{x,x_1,x_2\}$, on $2(i+j)-3$ vertices $\Longrightarrow$ We can also show again that $G'-V(P(3))$ has two disjoint $K_{1,2}$'s



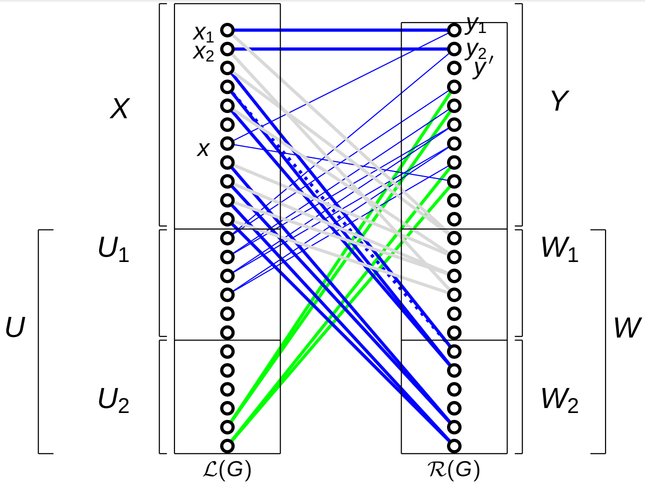
The graph G' has a path on $2(i+j-1)$ vertices and so has a path $P(3)$, that starts and ends in $X - \{x, x_1, x_2\}$, on $2(i+j) - 3$ vertices $\Longrightarrow$ We can also show again that $G' - V(P(3))$ has two disjoint $K_{1,2}$'s



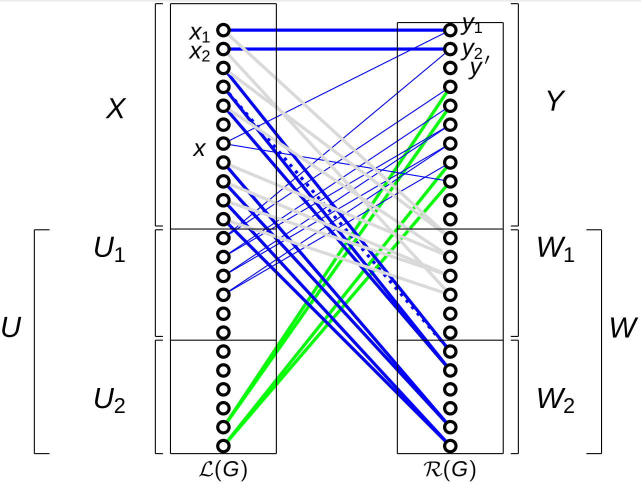
The graph G' has a path on $2(i+j-1)$ vertices and so has a path $P(3)$, that starts and ends in $X - \{x, x_1, x_2\}$, on $2(i+j) - 3$ vertices $\Longrightarrow$ We can also show again that $G' - V(P(3))$ has two disjoint $K_{1,2}$'s



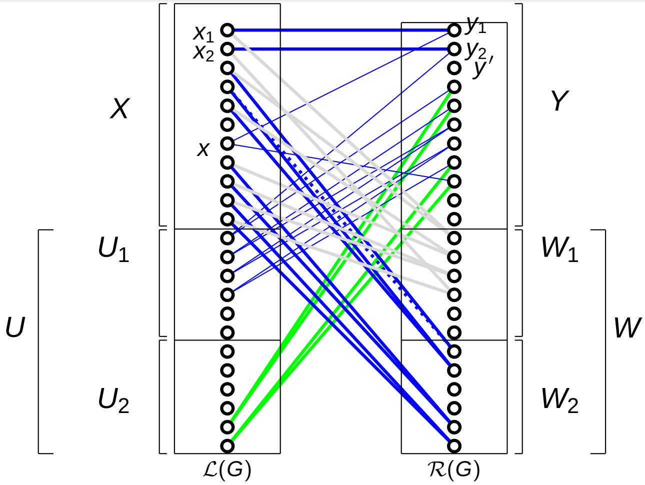
The graph G' has a path on $2(i+j-1)$ vertices and so has a path $P(3)$, that starts and ends in $X - \{x, x_1, x_2\}$, on $2(i+j) - 3$ vertices $\Longrightarrow$ We can also show again that $G' - V(P(3))$ has two disjoint $K_{1,2}$'s



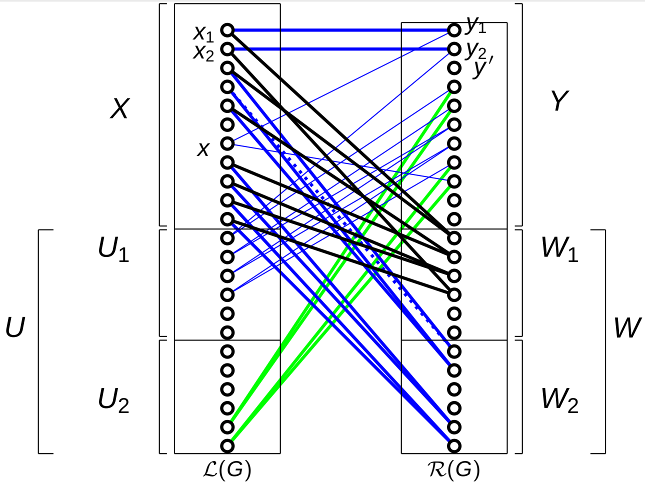
By Lemma there exists an $x_1 - x_2$ path P on
$2|W_1| - 1 + 2k + n(P(3)) - 1 = 2(s/2 - j - 1) - 1 + 2.2 + 2(i + j) - 3 - 1 = s + 2i - 3$
$\Longrightarrow$ The sequence of vertices P, P' form a cycle on $s + 2i - 3 + s - 2i + 3 = 2s$ vertices



By Lemma there exists an $x_1 - x_2$ path P on
$2|W_1| - 1 + 2k + n(P(3)) - 1 = 2(s/2 - j - 1) - 1 + 2.2 + 2(i + j) - 3 - 1 = s + 2i - 3$
$\Longrightarrow$ The sequence of vertices P, P' form a cycle on $s + 2i - 3 + s - 2i + 3 = 2s$ vertices



By Lemma there exists an $x_1 - x_2$ path P on $2|W_1| - 1 + 2k + n(P(3)) - 1 = 2(s/2 - j - 1) - 1 + 2.2 + 2(i + j) - 3 - 1 = s + 2i - 3$
$\Longrightarrow$ The sequence of vertices P, P' form a cycle on $s + 2i - 3 + s - 2i + 3 = 2s$ vertices



By Lemma there exists an $x_1 - x_2$ path P on
$2|W_1| - 1 + 2k + n(P(3)) - 1 = 2(s/2 - j - 1) - 1 + 2.2 + 2(i + j) - 3 - 1 = s + 2i - 3$
$\Longrightarrow$ The sequence of vertices P, P' form a cycle on $s + 2i - 3 + s - 2i + 3 = 2s$ vertices