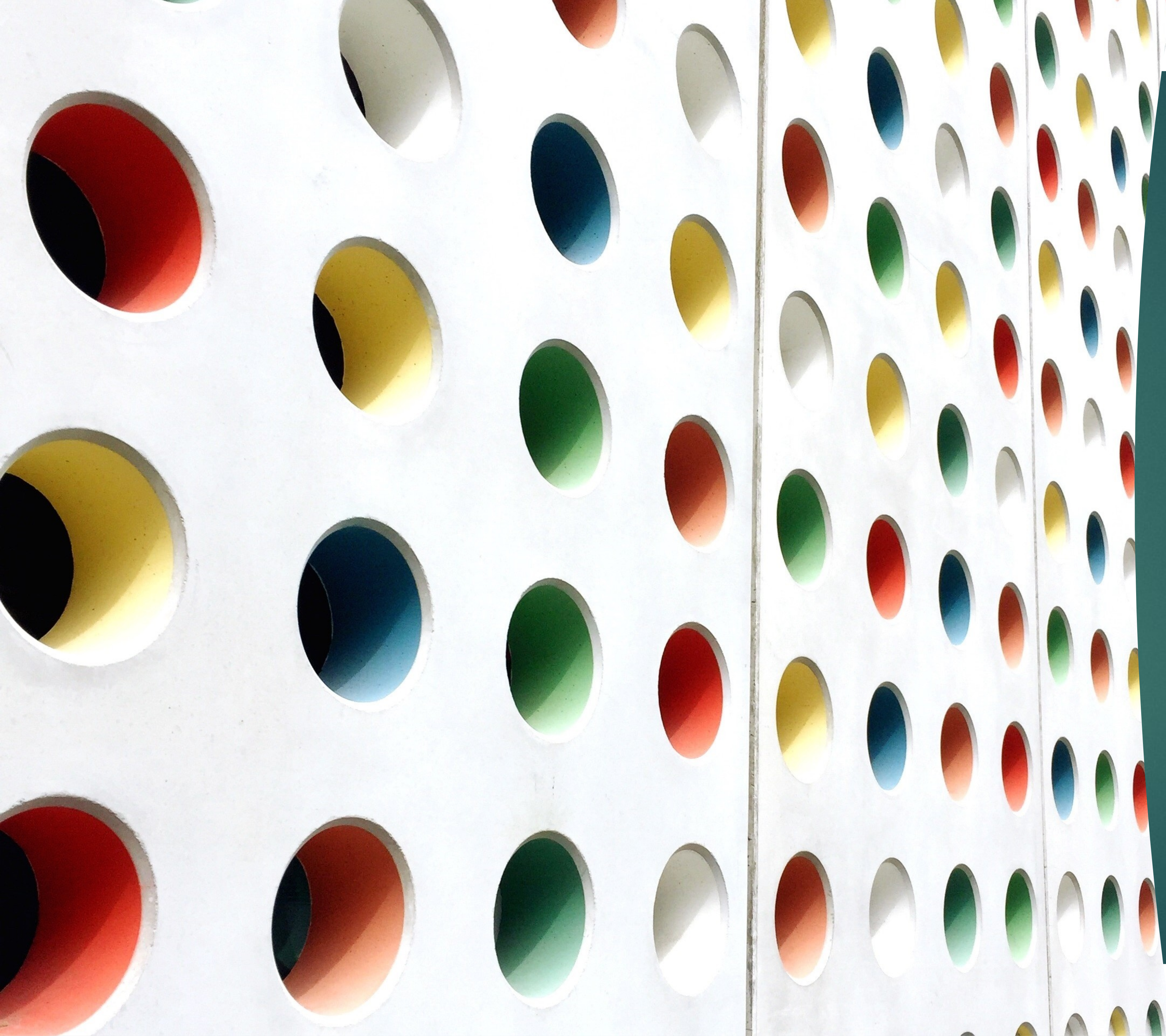


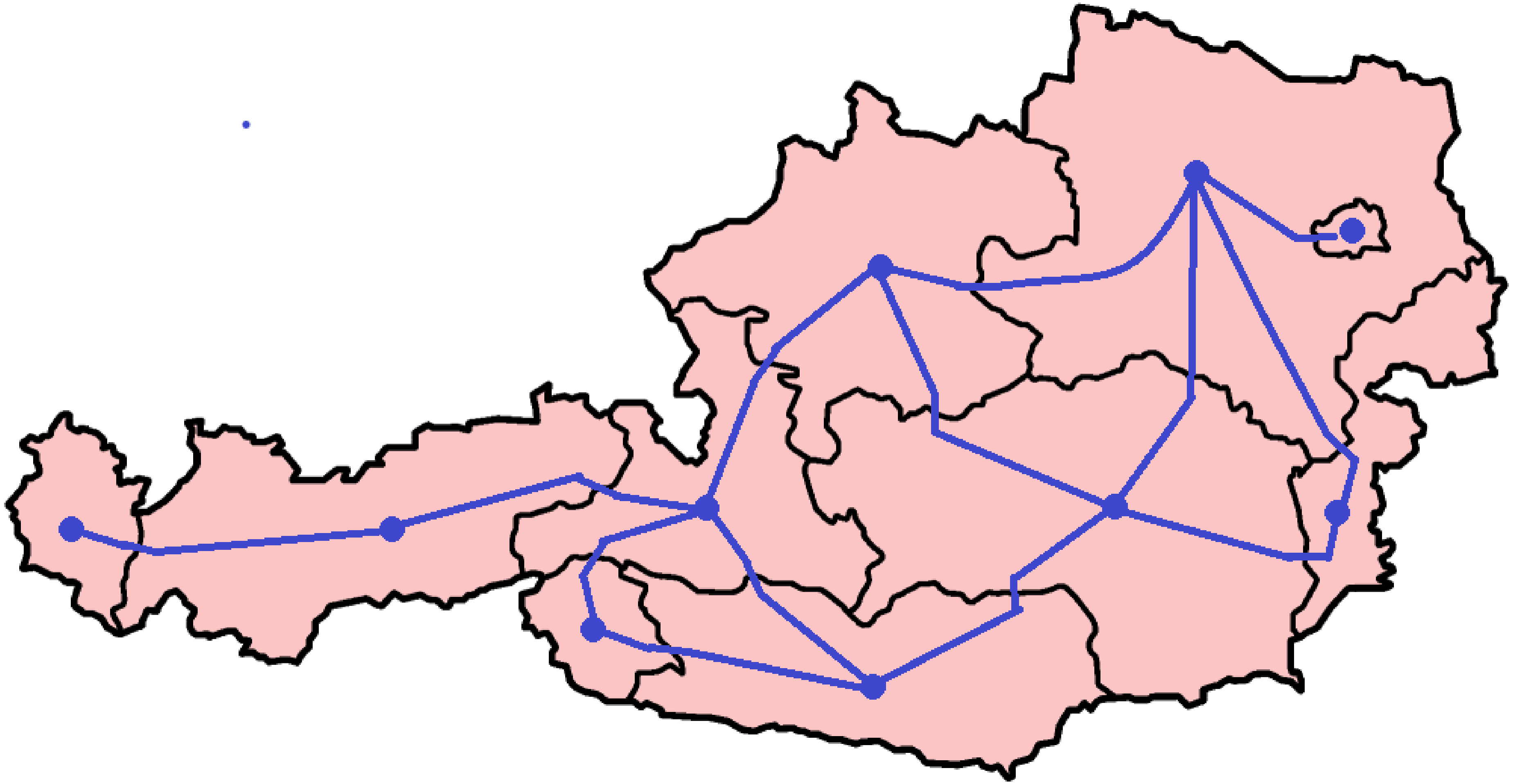
The Packing Chromatic Number of a graph

E Jonck
Sams 2022

This talk is dedicated to my collaborators and students.

The origin of colorings - The Four-Color Theorem

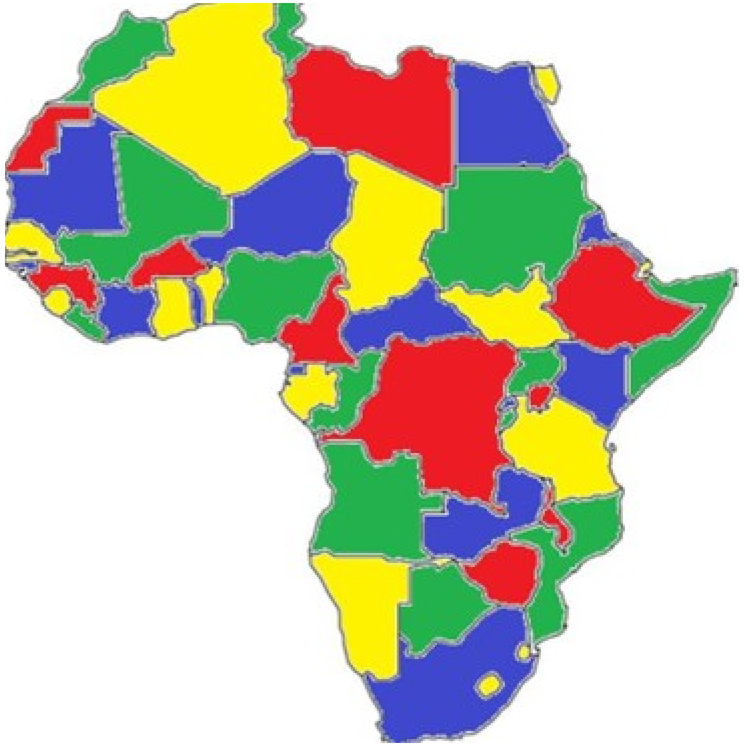
It all started with the coloring of maps. Maps are examples of planar graphs. While trying to color a map of the counties of England, Francis Guthrie formulated the four color conjecture, noting that four colors were sufficient to color a map so that no regions sharing a common border received the same color.



Guthrie's brother Frederik passed the question on to his mathematics professor Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton in 1852.

Arthur Cayley raised the problem at a meeting of the London Mathematical Society in 1879.

The same year, Alfred Kempe published a paper that claimed to establish the result, and for a decade the four color problem was considered solved.



For his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society.

In 1890, Heawood wrote in a paper that Kempe's argument was wrong. In that paper he proved the five color theorem, saying that every planar map can be colored with at most *five* colors, using ideas of Kempe.

In the following century, a vast amount of work and theories were developed to reduce the number of colors to four.

The four color theorem was finally proved in 1976 by Kenneth Appel and Wolfgang Haken. The proof went back to the ideas of Heawood and Kempe.

The four color theorem is also noteworthy for being the first major computer-aided proof.

Graph coloring is used in various research areas of computer science such **data mining**, **image segmentation**, **clustering**, **image capturing**, **networking** etc.

Completed PhD's Children

Chuck Garner, Samantha Dorfling, Ernst Joubert, Charl Ras, Riëtte Eiselen, Sonwabile Mafunda and Ronnie Maartens

Profs Parents

G Domke, I Broere, J Hattingh, P Dankelmann and E Jonck

Completed MSc's Children

S Dorfling, N Botha,**
Y Immelmann*, C Lochner*,
S Marcon, A Marcon,
R Maartens*, D Lessing*, O Nkuna*,
K Mashia, A Nxweni, J Sarila,
A Tivane, L Mabusela*, T Pramjeeth

* cum laude

Profs Parents

I Broere, P Dankelmann, R Maartens, T van Zyl and E Jonck

Current PhD's and MSc's

Andile, Oscar and De Villiers

Sympathy, David, Tshepo, Thandukwazi, Muzi, Tarryn, Sharon and Kim (to submit)
Lethlo (to submit)

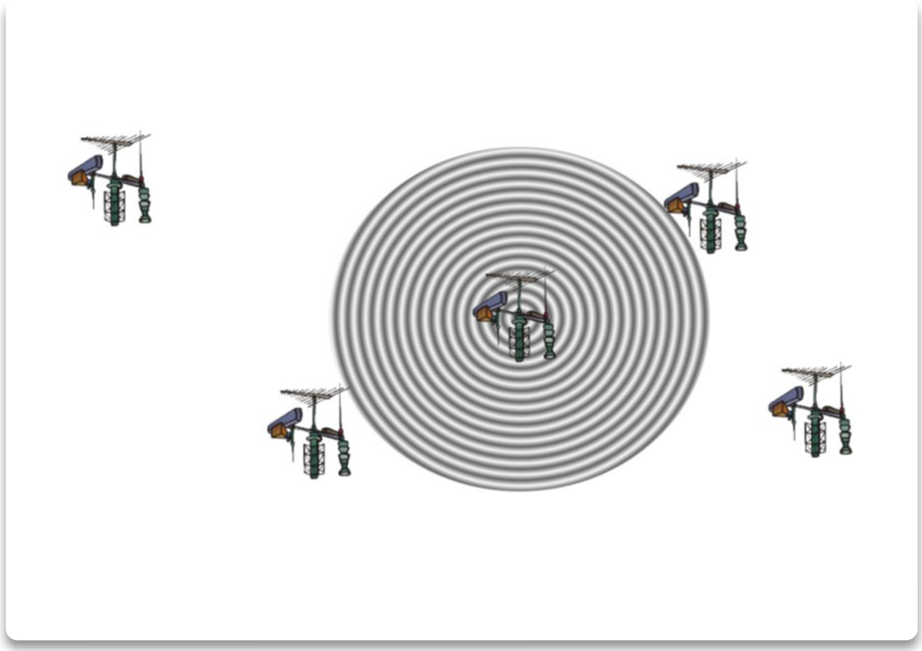
Post Doc

Alex Alochukwu

Profs

J Hattingh, Y Hardy, S Mukwembi, R Maartens, E Jonck

The packing chromatic number of a graph



Packing coloring originated from planning done in the broadcast industry to avoid interference of frequencies of different wireless radio stations.

The United States Federal Communications Commission formulated regulations with regards to the assignment of broadcast frequencies to radio stations. Two radio stations which have received the same broadcast frequency, must be located sufficiently far apart so that the broadcast frequencies do not interfere with each other.

- Goddard et al introduced the concept as the broadcast coloring in 2008.

Complexity of PCN of a graph

Theorem (Goddard, Hedetniemi, Hedetniemi, Harris, Rall '08)

Let G be a graph.

To decide if $\chi_\rho(G) \leq$ k is NP-complete (k on input).

To decide if $\chi_\rho(G) \leq 3$ is in P.

To decide if $\chi_\rho(G) \leq 4$ is NP-complete.

Theorem (Fiala, Golovach '09)

Decide if $\chi_\rho(G) \leq$ k for trees is NP-complete (k on input).

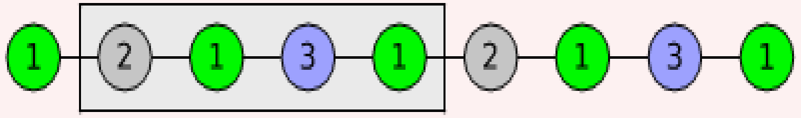
Definition d-packing

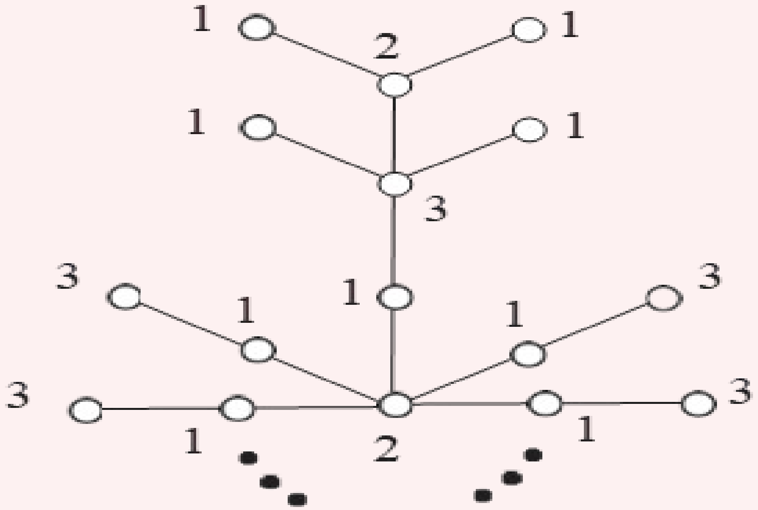
Graph G = (V, E), Pd ⊆ V is d-packing

if

∀ u, v ∈ Pd : dist(u, v) > d.

1-packing is an independent set

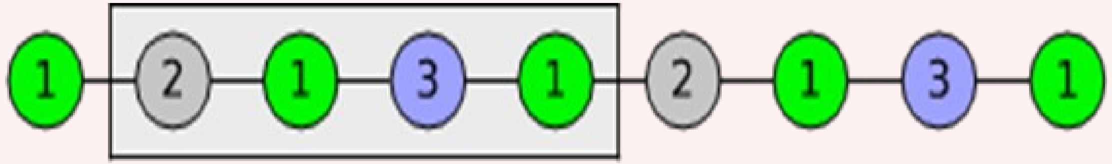


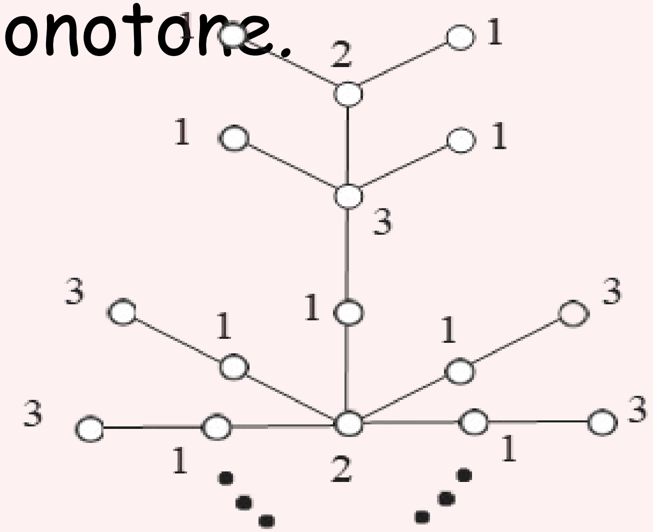


Definition PCN of a graph

The packing chromatic number is the minimum k such that $V = C_1 \cup C_2 \cup \ldots \cup C_k$,
denoted by $\chi_\rho(G)$.

If $c_1 \geq c_2 \geq \ldots \geq c_k$, then the coloring is monotone.

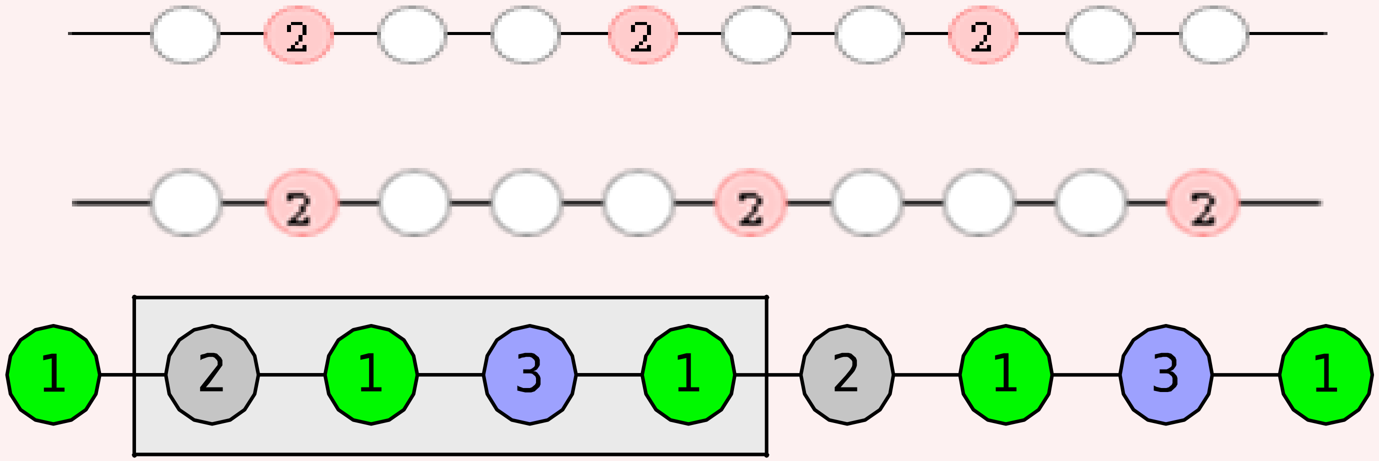




About xp(G)

Example of PCN of infinite path P∞

$xp(P_\infty) \le 3$



d-packing	ρd is density of d -packing
1	1/2
2	1/3 … 1/4
3	1/4

Note that

1/2 + 1/4 = 3/4

and

1/2 + 1/3 = 5/6

and

1/4 + 1/3 = 7/12

$\chi_\rho(P_\infty) \geq 3$

P_∞ is not 2 packing colorable.

Theorem

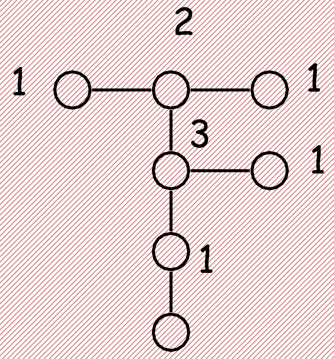
For any graph G and any m, where $m \leq \chi\rho(G)/2$, there exists a $\chi\rho(G)$-coloring $c: V(G) \rightarrow \{1, \dots, k\}$ such that $c_m \geq c_n$ for all $n \geq 2m$.

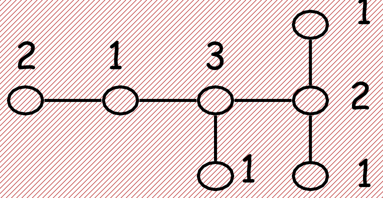
$n \geq 2m$	$m \leq X\rho(G)/2$,	2m
$c1 \geq c2, c3, \dots$	1	2
$c2 \geq c4, c5, \dots$	2	4
$c3 \geq c6, c7, \dots$	3	6
$c4 \geq c8, c8, \dots$	4	8
	… k/2	

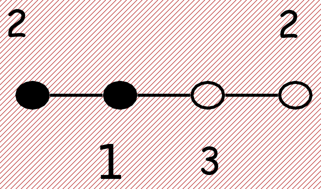
On uniquely packable trees

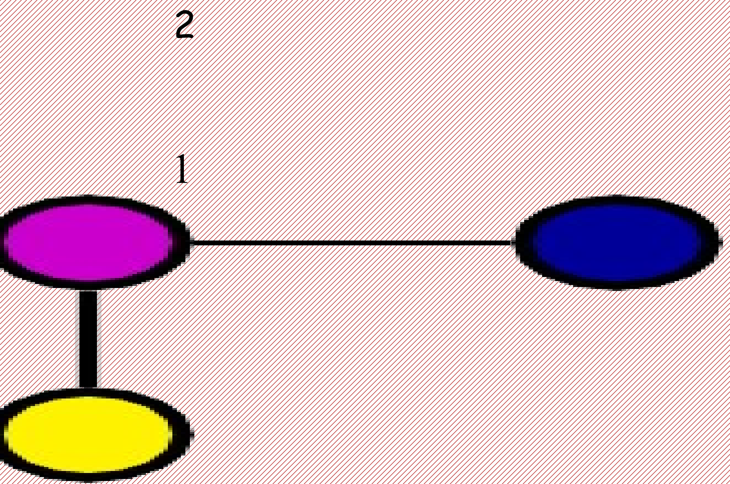
A. Alochukwu[*,†,⋆], M. Dorfling[‡], and E. Jonck[†]

F_1, F_2, F_3









Observation

Let v be a 1-vertex in a k-χ_ρ-packing of G, then $\deg(v) \le k - 1$.

Lemma

If G uniquely 3-χ_ρ-packable, then

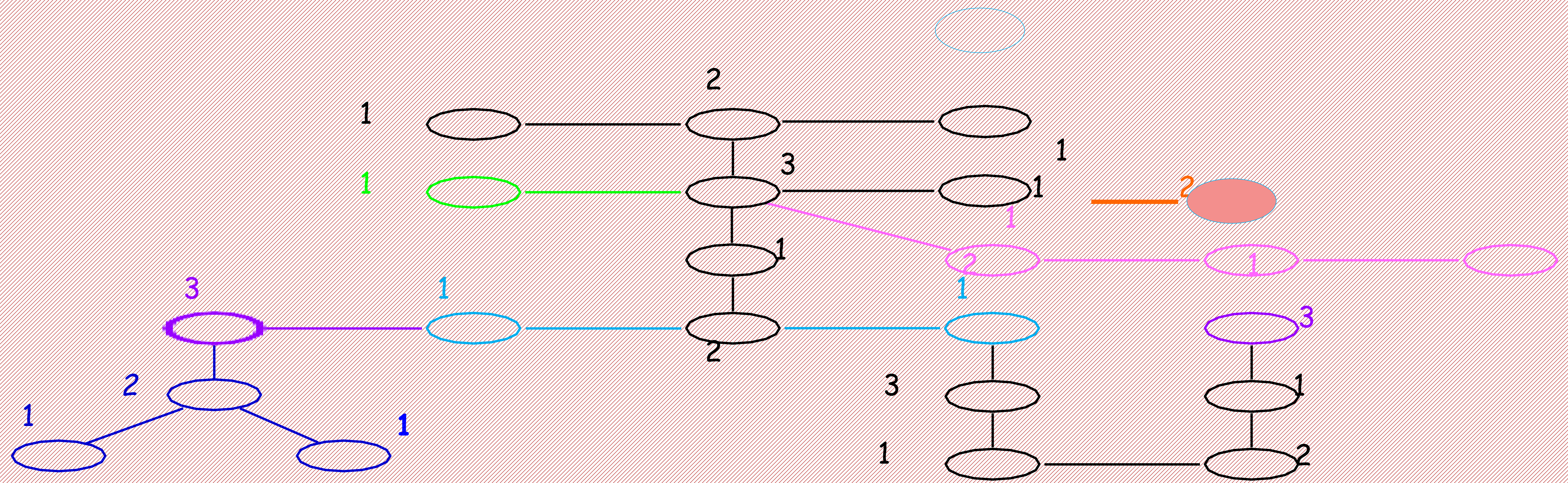
G has a 2-vertex adjacent to a 3-vertex.

O_1 O_2 O_3 O_4

O_5 O_6 O_7

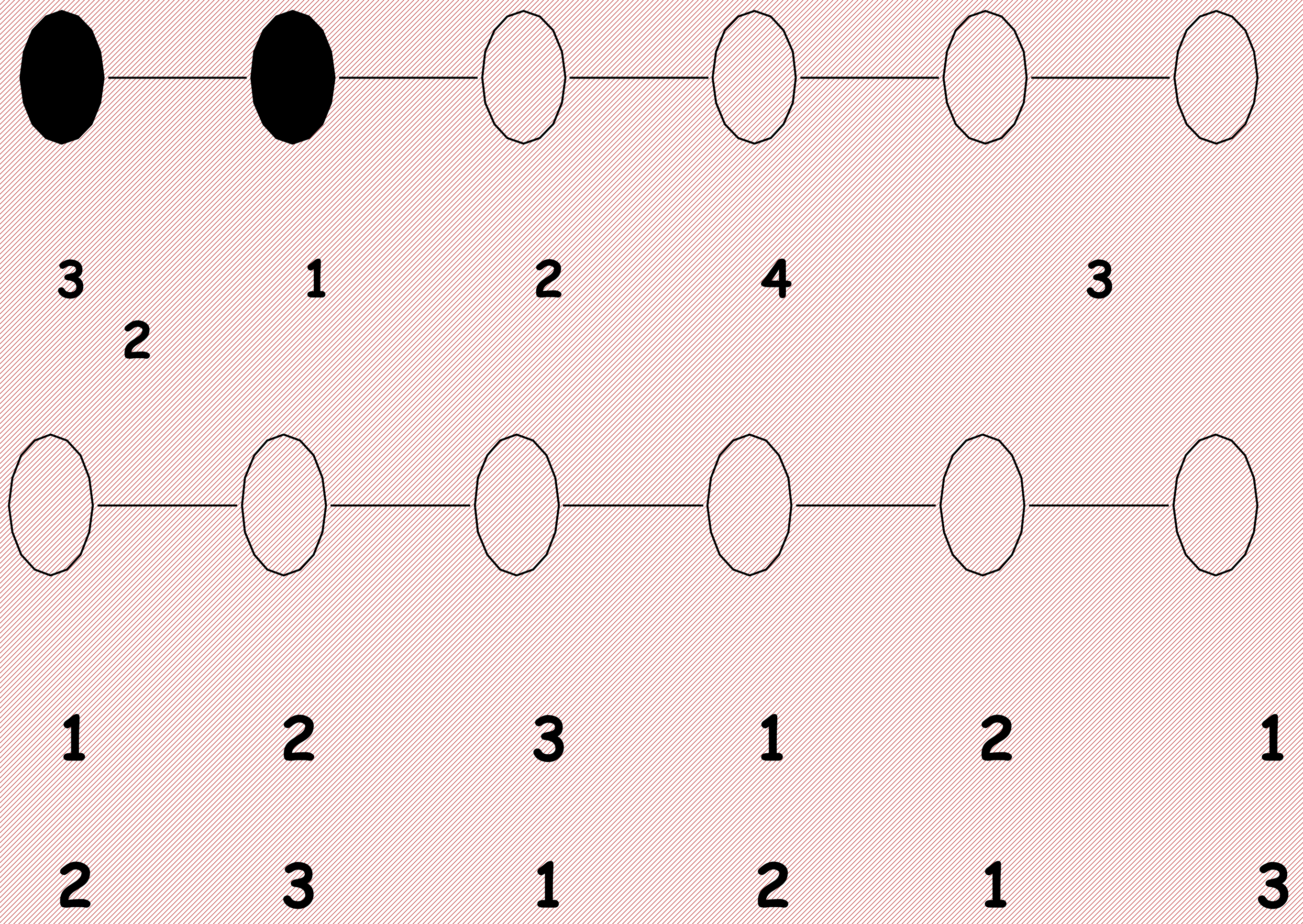
<u>Theorem</u>

T uniquely 3-χ_ρ-packable iff T obtained from F_1, F_2 or F_3 by O_i (repeatedly)



On uniquely 4- packable trees

A. Alochukwu, M. Dorfling, E Jonck, S. Mukwembi



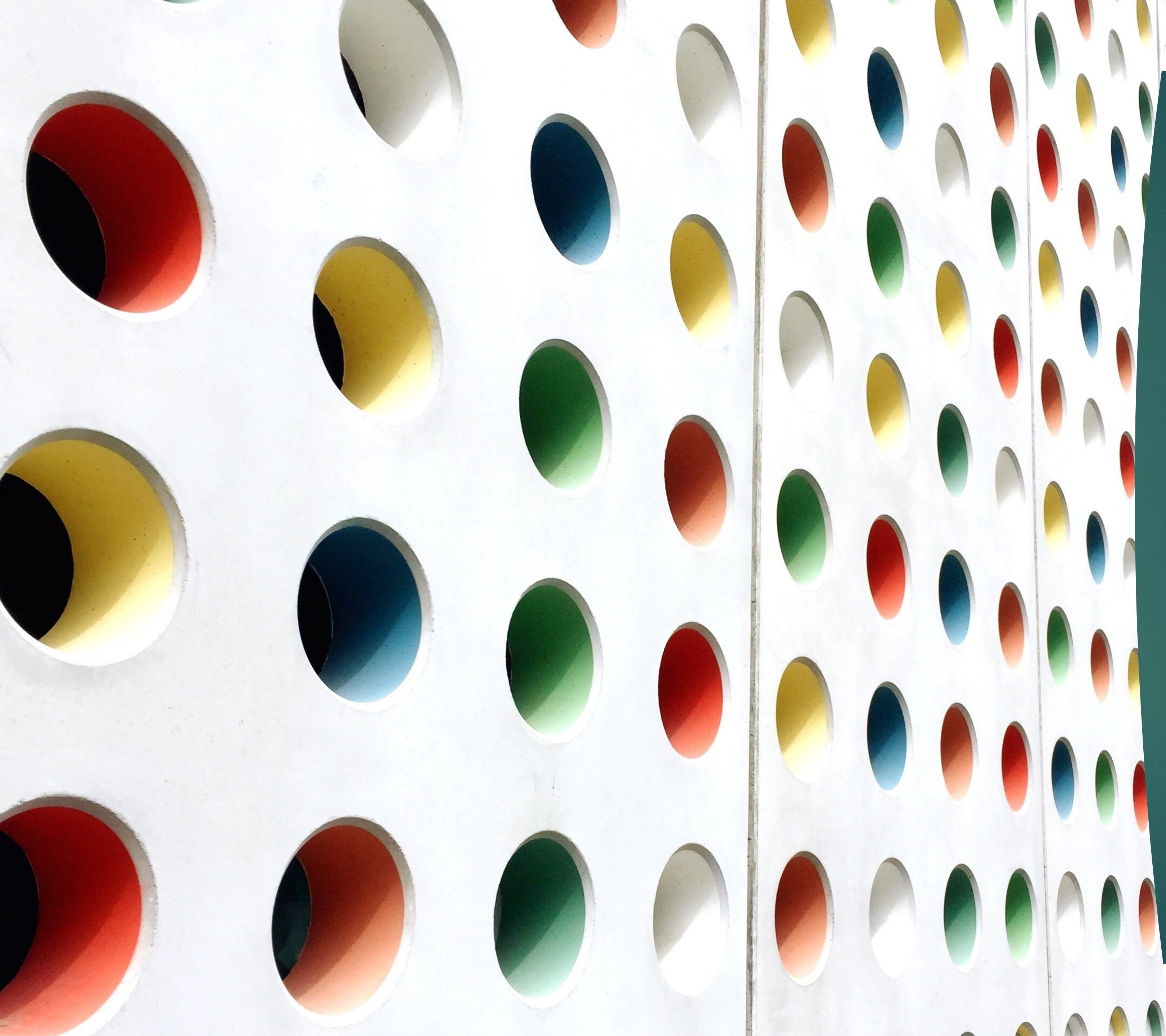
(1,1,2,3)-Colorings of Subcubic Graphs

P. Dankelmann, E. Jonck, R.J. Maartens, O. Nkuna

The *S*-packing number is a generalization of the packing chromatic number.

Gastineau and Togni showed that every subcubic graph is (1,1,2,2,2)-colorable. They

asked if any subcubic graph, except the Petersen graph, is (1,1,2,3)-colorable.

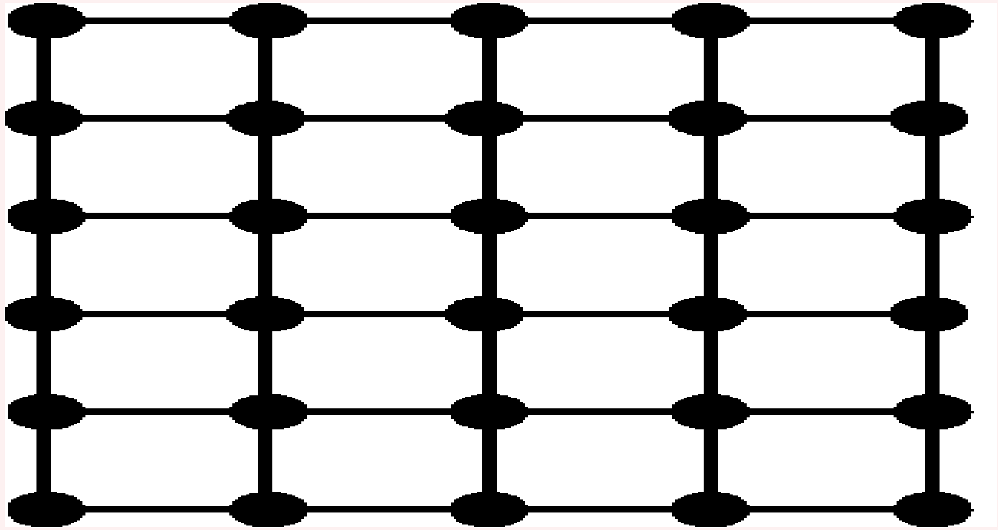


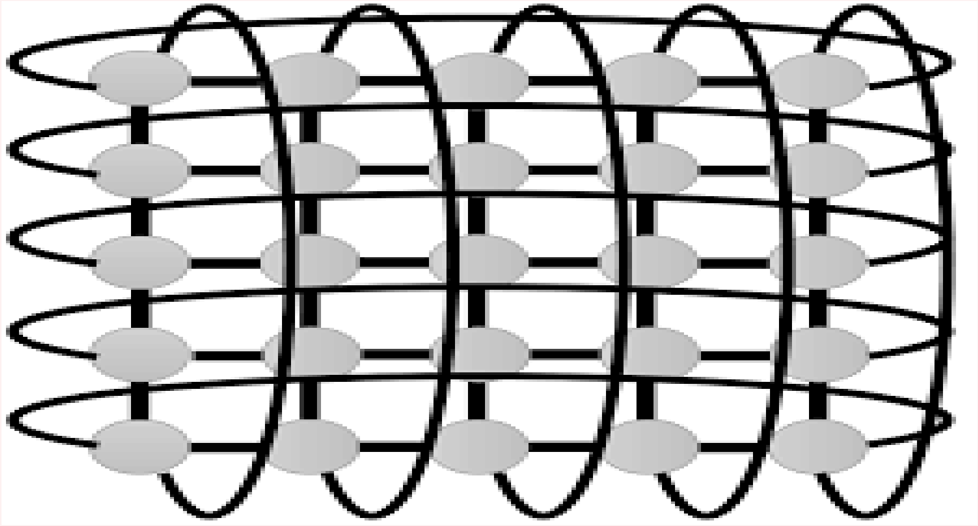
The relationship between the PCNs of the grid and the torus

J Hattingh, E Jonck,
D Lessing

We study the PCN of the grid and the torus

$P_r \square P_k$ $C_r \square C_k$





Square lattice / grid

Theorem (Goddard et al. 2008)
For infinite planar square lattice $P_r \square P_k$:
$9 \le \chi_\rho(P_r \square P_k) \le 23$, r and k tend to ± infinity.

Theorem (Communication Goddard, Schwenk 2008)
$\chi_\rho(P_r \square P_k) \le 22$, r and k tend to ± infinity.

Theorem (Fiala, Klavžar, Ludicky, 2009)
$10 \le \chi_\rho(P_r \square P_k)$, r and k tend to ± infinity
finity.

Theorem (Holub, Soukal, 2009)
$\chi_\rho(P_r \square P_k) \le 17$, r and k tend to ± infinity.
SAT-solver

Square Lattice

Theorem (Ekstein, Holub, Fiala, Ludicky, 2010)

$12 \le \chi_\rho(P_r \square P_k)$, r and k tend to ± infinity.

Theorem (Martin, Raimonde, Chen, Martin, 2016)

$\chi_\rho(P_r \square P_k)) \le 16$, r and k tend to ± infinity.
(Jonck, Dorfling too)

Theorem (Martin, Raimonde, Chen, Martin, 2017)

$13 \le \chi_\rho(P_r \square P_k)) \le 15$, r and k tend to ± infinity.

Theorem (Heule, Subercaseaux, 2022)

$\chi_\rho (P_r \square P_k) \ge 14$, r and k tend to ± infinity.

The torus

Jacobs, Jonck, and Joubert (2013) studied the packing chromatic number of the Cartesian product of cycles of specific length.

More precisely, we proved that

Case 1: if x ≥ 3, then $9 \leq \chi_\rho(C4 \square C4x) \leq 11$.

Moreover,

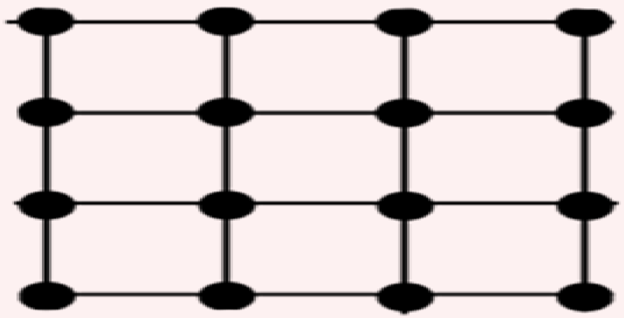
Case 2: if in addition x is divisible by 4, then $\chi_\rho(C4 \square Cxt) = 9$

Theorem Upperbound

Suppose G is a graph C4 □ Cq where q = 4n, $n \geq 3$. Then

$\chi_\rho(G) \leq 9$ if $n = 4x$; $x \geq 3$

$\chi_\rho(G) \leq 11$ if $n = 4x + 1$, $x \geq 1$ or $n=4x + 2$; $x \geq 1$ or $n=4x + 3$; $x \geq 0$



2	1	3	1
1		1	
3	1	2	1
1		1	

Basic block

Case 1

2	1	3	1
1	5	1	6
3	1	2	1
1	4	1	7

Block 1

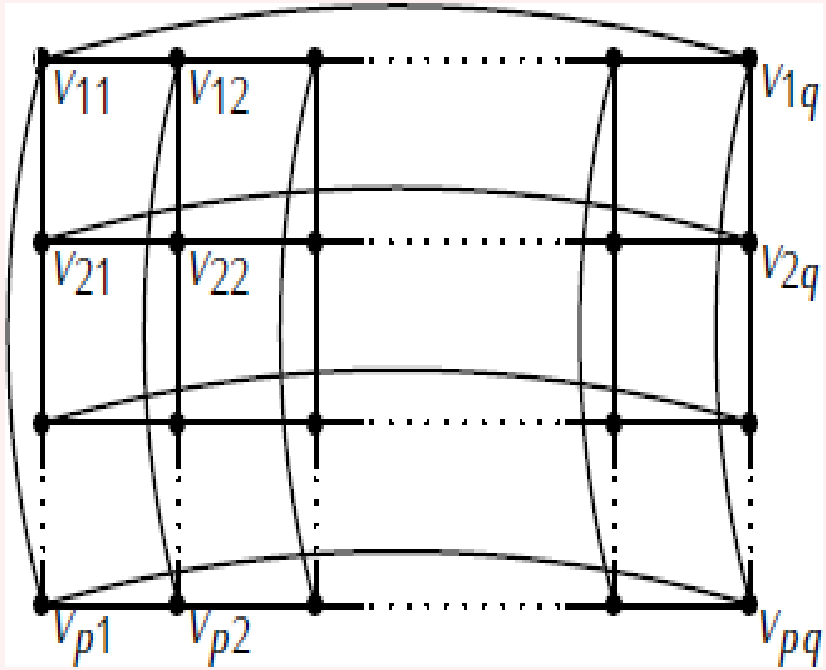
2	1	3	1
1	5	1	8
3	1	2	1
1	4	1	9

Block 2

2	1	3	1
1	5	1	9
3	1	2	1
1	4	1	8

Block2a

Define $q = 4n$, where n is the number of 4×4 blocks.

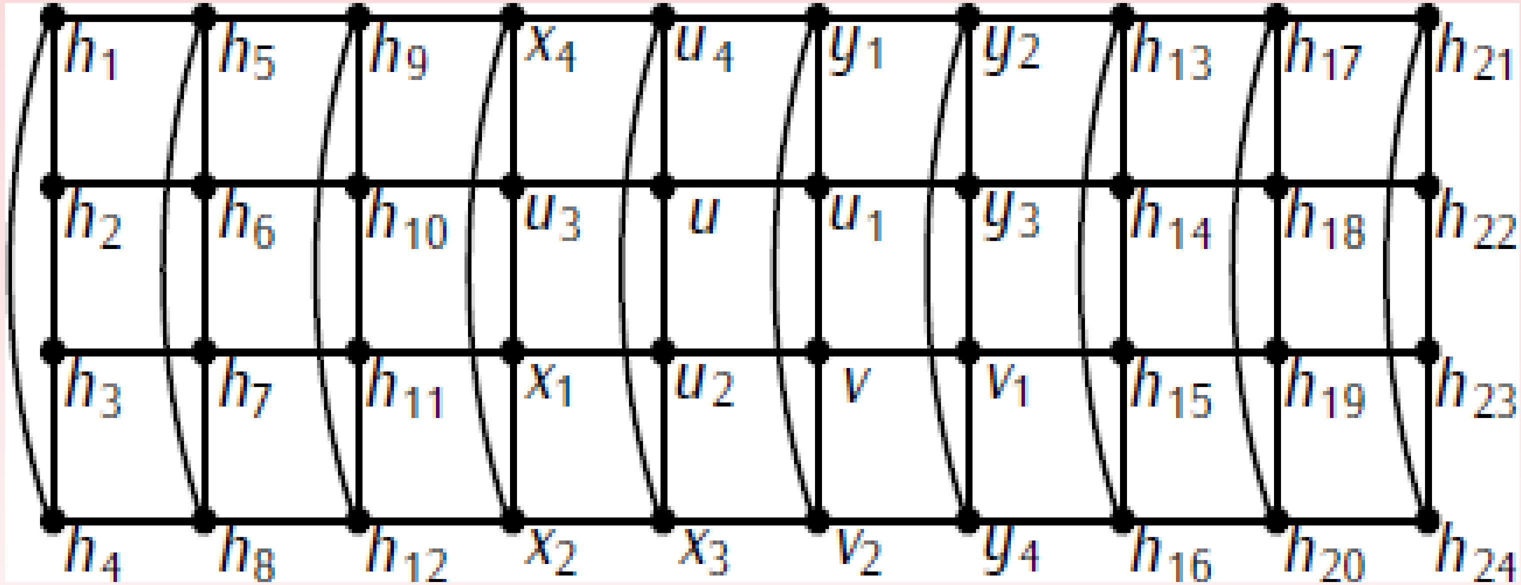


Sequence 1, 2, 1, 2a a total x times to create the pattern
1, 2, 1, 2a, 1, 2, 1, 2a,. . . 1, 2, 1, 2a.

Theorem

Suppose $G = C_4 \ C_q$, where $q = 4x$ and integer x, where $x \geq 3$. Then $\chi_\rho(G) \geq 9$.

Theoretical proof using



A comparison between the packing chromatic numbers of the grid and the torus

Conjecture 1

The packing chromatic number of the infinite square lattice is 15.

Also

Since $C_r \square C_k$ contains $P_r \square P_k$ where r and k tend to ± infinity,

$\chi_\rho (C_r \square C_k) \geq \chi_\rho (P_r \square P_k)$

by monotonicity of graphs w.r.t. coloring.

Conjecture 1 implies

$\chi_\rho (C_r \square C_k) \geq 15$

A 15-packing colored 72 x 72 tile was used to periodically color the infinite grid.

The same tile can be used to color the torus $C_r \square C_k$ where r and k tend to ± infinity.

Hence we have

$\chi_\rho (C_r \square C_k) \leq 15.$

Conjecture 2

The packing chromatic number of the torus $C_r \square C_k$ where r and k tend to ± infinity, is 15

Application

An integer linear programming model and a satisfiability test model for the packing coloring problem of graphs are developed. The proposed models outperform other exact methods such as a back-tracking and dynamic algorithm. In particular, the packing chromatic numbers and improved bounds have been found for the Cartesian products of paths and cycles.

Satisfiability and SAT solver

Satisfiability problem

The Satisfiability problem is to decide,
given a SAT formula,
whether it is satisfiable (or consistent) or not.

Sometimes, if the SAT formula is satisfiable/consistent,
we would also like to compute a satisfying assignment (or model).

Reference

Discussiones Mathematicae
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sciendo

A SURVEY ON PACKING COLORINGS

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