Eigenvalues of small rank parametric perturbations of H-expansive matrices

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Indefinite inner product space

- Function $[.,.]$ from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ is called an indefinite inner product in $\mathbb{C}^n$ if:
  - Linear in the first argument
  - Antisymmetric
  - nondegenerate, i.e., if $[x, y] = 0 \, \forall y \in \mathbb{C}^n$ then $x = 0$
- Only exception from the standard inner product is that $[x, x]$ may be nonpositive for $n \neq 0$.
- For every $n \times n$ Hermitian matrix H, i.e., $H = H^*$

$$[x, y] = \langle Hx, y \rangle, \quad x, y \in \mathbb{C}^n$$

  determines an inner product on $\mathbb{C}^n$.
- Converse also holds.

Jordan form

A matrix A in (real or complex) Jordan form is expressed as the sum of Jordan segments, that is,

$$A = J(\lambda_1) \oplus \ldots \oplus J(\lambda_p) \quad \text{or} \quad A = J(\gamma_1) \oplus \ldots \oplus J(\gamma_p).$$

Each Jordan segment consists of blocks of the form

$$J_n(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \\ 0 & \lambda_i & 1 & 0 & \ldots \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix} \text{ and } J_n(\gamma_i) = \begin{bmatrix} \gamma_i & I_2 & 0 & \ldots & \\ 0 & \gamma_i & I_2 & 0 & \ldots \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & \gamma_i \end{bmatrix},$$

where $\gamma = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for a complex eigenvalue $\lambda = a \pm ib$.

Different classes for the matrix pair (A, H)

Let A be an $n \times n$ matrix, and let $H = H^*$ and invertible of size $n \times n$. Then A is called:

$$\begin{array}{ll} H-expansive & \text{when } A^*HA - H \geq 0; \\ H-contractive & \text{when } A^*HA - H \leq 0; \\ H-unitary & \text{when } A^*HA - H = 0. \end{array}$$

For the last class canonical forms exist.
For the first two classes, only simple forms exist.

Goal: Interested in eigenvalues of small rank perturbations within these classes.

Canonical forms and Simple forms for (A, H)

Let A be in Jordan canonical form, i.e., $A = J_n(1) \oplus J_n(1)$, when A only has eigenvalue 1 with n even. Then the canonical form for the corresponding matrix H when A is H-unitary is:

$$H = \begin{bmatrix} 0 & 0 & 0 & Q_n \\ 0 & 0 & -Q_n^T & 0 \\ 0 & -Q_n & 0 & 0 \\ Q_n^T & 0 & 0 & 0 \end{bmatrix},$$

where Q_n is a $\frac{n}{2} \times \frac{n}{2}$ matrix with know entries.

If A is H-expansive only a simple form for matrix H exists. Here H is for example

$$H = c \left[\begin{array}{cccc|cccc} 0 & \cdots & & & & & 0 & 1 \\ 0 & & & & & & \iddots & \\ \vdots & & & & & & & \\ & & & & & (-1)^{\frac{n-4}{2}} & * & \\ & & & & (-1)^{\frac{n-2}{2}} & & & \\ \hline & & & (-1)^{\frac{n-2}{2}} & & & & \\ \vdots & & (-1)^{\frac{n-4}{2}} & & & & & \\ & & & & & * & & \\ 0 & \iddots & & * & & & & \\ 1 & & & & & & & \end{array} \right].$$

Low rank perturbation

Consider the example where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & 0 \end{bmatrix}.$$

The eigenvalue of A is $\lambda = 0$ with multiplicity 3.
However, the characteristic polynomial of B is given by $p_B(\lambda) = \lambda^3 - \varepsilon$, so the distinct eigenvalues of B are given by $\varepsilon^{1/3}\,\text{exp}^{2k\pi i/3}$ for $k = 0, 1, 2$. **Conclusions:**

- Rank of a matrix is a discontinuous function of its entries;
- Eigenvalues can completely change with small rank perturbations;
- Matlab is very sensitive when it comes to computations.

First observation

Proposition *Let A be H-unitary. Then a perturbation $B = A + UV^*A = (I + UV^*)A$, with U and V both of size $n \times k$ and rank k is also H-unitary if and only if $V = HUE$ where E is an invertible $k \times k$ matrix satisfying*

$$U^*HU = -(E^{-1} + (E^*)^{-1}) = -E^{-1}(E + E^*)(E^*)^{-1}. \tag{1}$$

Proof.

$$B^*HB = A^*HA + A^*(VU^*H + HUV^* + VU^*HUV^*)A$$

This is equal to H if and only if

$$V(U^*H + U^*HUV^*) = -HUV^*.$$

Both sides have rank k implies there is an invertible $k \times k$ matrix E such that $V = HUE$.

$$HUEU^*H + HUE^*U^*H + HUEU^*HUE^*U^*H = 0.$$

Now, H is invertible, and $LU = I_k$, thus

$$E + E^* + E(U^*HU)E^* = 0.$$

$\square$

Second observation

Proposition *Let A and H be real $n \times n$ matrices, such that $H = H^T$ is invertible and A is H-orthogonal, and let U be an $n \times k$ real matrix and E be a $k \times k$ real matrix such that (1) holds. Let G be a skew-symmetric real $k \times k$ matrix such that $E = -2(U^T HU - 2G)^{-1}$. Consider the matrix $B = (I - 2U(U^* HU + 2G)^{-1} U^* H)A$. Then*

$$\det B = (-1)^k \det A.$$

For odd k the H-orthogonal matrix B is not in the same connected component as the matrix A, while for even k the matrix B is in the same connected component as the matrix A.

Rank k perturbation vs k consecutive rank 1 perturbations

Proposition *Let A be H-expansive, U_1 an $n \times k$ matrix, and U_2 an $n \times l$ matrix. Let G_1 and G_2 be skew-Hermitian such that $U_i^* H U_i + 2G_i$ is invertible for $i = 1, 2$. Form the rank k perturbation*

$$B_1 = (I - 2U_1(U_1^* H U_1 + 2G_1)^{-1} U_1^* H)A,$$

and consider the rank l perturbation of B_1 given by

$$B_2 = (I - 2U_2(U_2^* H U_2 + 2G_2)^{-1} U_2^* H)B_1.$$

Then B_2 can be viewed as a rank $k + l$ perturbation of A, given by

$$B_2 = \left(I - 2 \begin{bmatrix} U_2 & U_1 \end{bmatrix} \left(\begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \begin{bmatrix} U_2 & U_1 \end{bmatrix} + \begin{bmatrix} 2G_2 & U_2^* H U_1 \\ -U_1^* H U_2 & 2G_1 \end{bmatrix} \right)^{-1} \begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \right) A.$$

Proposition *Let A be H-expansive, and let $B(t) = (I + tUE^*U^*H)A$, where E satisfies (1). Then $B(t)$ is also H-expansive in one of the following two cases:*

a. *U^*HU is positive semidefinite and either $t \leq 0$ or $t \geq 1$,*

b. *U^*HU is negative semidefinite and $0 \leq t \leq 1$.*

Proof. Computing $B(t)^*HB(t)$, we get

$$B(t)^*HB(t) = (A^* + tA^*HUEU^*)H(A + tUE^*U^*HA)$$
$$= A^*HA + (t - t^2)\langle A^*HU(E + E^*)U^*HA. \qquad (2)$$

Hence

$$B^*(t)HB(t) - H = A^*HA - H + (t - t^2)A^*HU(E + E^*)U^*HA$$

In particular, since A is H-expansive

$$B(t)^*HB(t) - H \geq (t - t^2)A^*HU(E + E^*)U^*HA. \qquad (3)$$

Recall, (1) the signatures of U^*HU and $-E - E^*$ coincide. $\square$

Proposition *Let the $n \times n$ matrix A be H-expansive, and let U be an $n \times k$ matrix such that U^*HU is positive definite or negative definite. Let E be a $k \times k$ invertible matrix such that (1) is satisfied, and let $B(t) = (I + tUE^*U^*H)A$. Then the following hold:*

a. *In case U^*HU is positive definite then $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when* $0 < t \leq 1$

b. *In case U^*HU is negative definite then $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when either $t < 0$ or $t \geq 1$.*

H-unitary case

Next, we specialize to the case where A is H-unitary. The inequality becomes an equality:

$$B(t)^*HB(t) - H = (t - t^2)A^*HU(E + E^*)U^*HA.$$

We can sharpen an earlier proposition to the following:

Proposition *Let A be H-unitary, and let $B(t) = (I + tUE^*U^*H)A$, where E satisfies* (1). *Then $B(t)$ is H-expansive if and only if:*

U^*HU is positive semidefinite and either $t \leq 0$ or $t \geq 1$,

U^*HU is negative semidefinite and $0 \leq t \leq 1$.

Also, $B(t)$ is H-contractive if and only if:

U^*HU is positive semidefinite and $0 \leq t \leq 1$,

U^*HU is negative semidefinite and either $t \leq 0$ or $t \geq 1$.

In all other cases $B(t)^*HB(t) - H$ is indefinite.

Proposition *Let matrix A be H-unitary, and let U be an $n \times k$ matrix such that U^*HU is positive definite or negative definite. Let E be a $k \times k$ invertible matrix chosen as before, and let $B(t) = (I + tUE^*U^*H)A$. Then $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when $t = 1$.*

Eigenvalues cross the unit circle as t increases through $t = 1$

Proposition *Let A be H-unitary, and let $B(t) = (I + tUE^*U^*H)A$, where E is chosen as before. Suppose that for $t = 1$, $B(t)$ has an eigenvalue λ on the unit circle which is not an eigenvalue of A. Viewing λ as a function of t, we have the following two possibilities*

1. *λ crosses the unit circle from outside to inside if either $E + E^* > 0$ and $\langle Hx, x\rangle > 0$ or $E + E^* < 0$ and $\langle Hx, x\rangle < 0$,*

2. *λ crosses the unit circle from inside to outside if either $E + E^* > 0$ and $\langle Hx, x\rangle < 0$ or $E + E^* < 0$ and $\langle Hx, x\rangle > 0$.*

Examples

Example 1: Let $A = J_2(1) \oplus J_2(1) \oplus J_2(1) \oplus J_2(1)$ and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

A is H-expansive, i.e., $A^*HA - H \geq 0$;
First two Jordan blocks are "coupled";
Last two blocks are not coupled.

For a real u randomly generated in Matlab (u=randn(8,1)), we see that:
Generically B will have 3 blocks of size 2 at eigenvalues 1,
Two real eigenvalues not equal to one.
Conclusion: the rank one perturbation has a preference to destroy the uncoupled block.

Example 2: Let $A = J_2(1) \oplus J_2(1) \oplus J_2(1) \oplus J_2(1)$ and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

If we would take very specific vectors u, namely those with the last four entries zero, then something else happen:

There are two Jordan blocks with size 2 at eigenvalue 1
One Jordan block of size 3 at eigenvalue 1, and also an eigenvalue at -1.
Conclusion: this is because then we only make the perturbation to the coupled blocks.

References

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Thank you for your attention

Let $A = J_2(1) \oplus J_2(1) \oplus J_2(1)$ and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A is H-expansive

For generic u there are 4 eigenvalues at 1, and two real eigenvalues, since the uncoupled block is destroyed and replaced by two real eigenvalues.

Product of these real eigenvalues is -1.

For non-generic case where $u_6 = 0$, the perturbed matrix $B = (I - \frac{2}{u^T H u} u u^T H)^{-1} A$ has five eigenvalues 1 and one eigenvalue equal to -1.

Compare the case where A is H-orthogonal.