A Toeplitz-like operator with rational symbol having poles on the unit circle

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Motivation

Let Ω an $m \times m$ rational matrix function with no poles on $\mathbb{T}$.

The block Toeplitz operator with symbol Ω is defined by

$$T_\Omega : H^p_m \to H^p_m,\ T_\Omega f = \mathbb{P}(\Omega f)$$

where $\mathbb{P}$ is the Riesz projection of $L^p_m(\mathbb{T})$ onto $H^p_m(\mathbb{T})$.

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The properties of interest in this study are

- T_Ω is bounded,
- T_Ω is Fredholm exactly when $\det \Omega(z) \neq 0$ for $z \in \mathbb{T}$,
- Wiener-Hopf factorization of Ω provide information on index of T_Ω, and on the kernel index of T_Ω.

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- $\Omega \notin L_m^\infty$ and so clearly T_Ω is not bounded,
- But does it follow that T_Ω is Fredholm $\iff \det \Omega(z) \neq 0$ for $z \in \mathbb{T}$?
- And is a Wiener-Hopf factorization of Ω possible that allows one to determine the index?

Ω has a pole on $\mathbb{T}$

Notation:

- Let Rat ($\mathrm{Rat}^{m\times m}$) be the set of rational functions ($m \times m$ rational matrix functions, respectively),
- $\mathrm{Rat}(\mathbb{T})$ rational functions with all its poles on $\mathbb{T}$ and $\mathrm{Rat}_0(\mathbb{T})$ the strictly proper rational functions with all its poles in $\mathbb{T}$.
- By $\mathcal{P}$ we shall mean the space of all polynomials,
- $\mathcal{P}_n$ polynomials of degree at most n.

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We define $\mathrm{Rat}^{m\times m}(\mathbb{T}), \mathrm{Rat}_0^{m\times m}(\mathbb{T}), \mathcal{P}^{m\times m}$ and $\mathcal{P}_n^{m\times m}$ similarly.

Ω has a pole on $\mathbb{T}$

Define the Toeplitz-type operator $T_\Omega \left(H^p_m(\mathbb{T}) \to H^p_m(\mathbb{T})\right)$ as:

$$\mathrm{Dom}(T_\Omega) = \left\{\varphi \in H^p_m(\mathbb{T}) : \Omega\varphi = g + \rho, g \in L^p_m(\mathbb{T}), \rho \in \mathrm{Rat}_0(\mathbb{T})\right\}$$
$$T_\Omega\varphi = \mathbb{P}g, \qquad \varphi \in \mathrm{Dom}(T_\Omega)$$

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H^2-symbol, $T^{HW}_\omega, \omega \in H^2$ (Hartmann - Wintner, 1950, 1954)
$\mathrm{Dom}(T^{HW}_\omega) = \{h \in H^2 : \omega h = g_1 + g_2 \in L^1, g_1 \in H^2, g_2 \perp H^2\}$,

$$T^{HW}_\omega h := g_1$$

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Bargmann-Segal space B, $\widetilde{T}_\omega$, ω entire function (Janas, 1990)
$\mathrm{Dom}(\widetilde{T}_\omega) = \{h \in B : \omega h = g + r, g \in B, \int r\overline{p}d\mu = 0, \forall p \in \mathcal{P}\}$,

$$\widetilde{T}_\omega h := g$$

Rest of the presentation

- Basic properties
- Factorization
- Fredholm properties

Basic Properties

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Let $\Omega \in \mathrm{Rat}^{m\times m}$, possibly with poles on $\mathbb{T}$. Then T_Ω is a well-defined, closed, densely defined linear operator on H^p_m. More specifically, $\mathcal{P}^m \subset \mathrm{Dom}\,(T_\Omega)$. Moreover,

$$S_- T_\Omega S_+ f = T_\Omega f \qquad \text{for all } f \in \mathrm{Dom}\,(T_\Omega),$$

where $S_+ = T_{zI_m}$ and $S_- = T_{z^{-1}I_m}$ on H^p_m.

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The fact that the domain contains the polynomials and that T_Ω satisfies a Brown-Halmos type identity means that the action of T_Ω can be represented by a block Toeplitz matrix.

Basic Properties

For $\Omega \in \mathrm{Rat}^{m \times m}$ with poles on the unit circle, $\mathbb{T}$, the action of T_Ω can be given by a block-Toeplitz matrix of the form

$$\begin{pmatrix} A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \ddots & \ddots & \ddots & \ddots & \end{pmatrix}$$

where $A_n, n \in \mathbb{Z}$ are $m \times m$ matrices.

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where $A_n, n \in \mathbb{Z}$ are $m \times m$ matrices. Here $A_n, n \in \mathbb{N}$ have the same properties as in the bounded case but there is a polynomial growth bound on $A_{-n}, n \in \mathbb{N}$.

Basic Properties: Domain

Domain in scalar case.

$$\omega = \frac{s}{q} \in \mathsf{Rat}(\mathbb{T}), \Rightarrow \mathrm{Dom}\,(T_\omega) = qH^p + \mathcal{P}_{\mathrm{degree}\, q-1}$$

How to extend this to matrix case?

Basic Properties: Domain

Let Ω be given by

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1m} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2m} \\ \vdots & & \ddots & \\ \omega_{m1} & \omega_{m2} & \cdots & \omega_{mm} \end{pmatrix}$$

with $\omega_{ij} = \frac{s_{ij}}{q_{ij}} \in \mathrm{Rat}(\mathbb{T})$. For $f = (f_1, f_2, \ldots f_m)^\intercal \in \mathrm{Dom}\,(T_\Omega)$ we need $f_j \in \mathrm{Dom}\,(T_{\omega_{ij}}), i = 1, \ldots, m$.

Given this, would

$$\mathrm{Dom}\,(T_\Omega) = qH^p_m + \mathcal{P}^m_{\mathrm{degree}\,q-1}$$

be the correct choice, where $q = \prod q_j$ for $q_j = LCM^m_{i=1}\{q_{ij}\}$?

Basic Properties: Domain

Two complications:

- For $q(z) = z^2 - 1 = (z+1)(z-1)$,
  $qH_2^p + \mathcal{P}_1^2 \neq ((z-1)H^p + \mathbb{C}) \oplus ((z+1)H^p + \mathbb{C})$.

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- For $\Omega(z) = \begin{pmatrix} 1 + \frac{1}{z-1} & \frac{1}{z-1} \\ \frac{1}{z+1} & 1 + \frac{1}{z+1} \end{pmatrix}$, $\det \Omega(z) = \frac{z^2+2z-1}{z^2-1} \not\equiv 0$ and $f = (f_1, -f_1)^T$ is in the domain of T_Ω for each $f_1 \in H^p$. Note that the LCM of the denominator for each of columns is $z^2 - 1$. But, even for $f_1 \notin \mathrm{Dom}\,(T_{\frac{1}{z^2-1}})$ we do have $\begin{pmatrix} f_1 \\ -f_1 \end{pmatrix} \in \mathrm{Dom}\,(T_\Omega)$.

Basic Properties: Domain

Let $\Omega \in \mathrm{Rat}^{m\times m}$ and write $\Omega = \Omega_1 + \Omega_2$ where $\Omega_1 \in L_\infty^{m\times m}$ and $\Omega_2 \in \mathrm{Rat}_0^{m\times m}(\mathbb{T})$. Then $\mathrm{Dom}\,(T_\Omega) = \mathrm{Dom}\,(T_{\Omega_2})$.

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Suppose $\Omega_2 = \left[\frac{s_{ij}}{q_{ij}}\right]$ with $q_{ij} \in \mathbb{T}$ and let $q = \prod q_j$ where $q_j = LCM_{i=1}^{m}\{q_{ij}\}$, i.e., the least common multiple of the denominators in the j-th column of Ω_2.

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$$qH_m^p + \mathcal{P}^m_{\mathrm{degree}\,q-1} \subset \oplus_j \left(q_j H^p + \mathcal{P}_{\mathrm{degree}\,q_j-1}\right) \subset \mathrm{Dom}\,(T_\omega)$$

and both inclusions can be strict.

Basic Properties: Domain

Let $\Omega \in \mathsf{Rat}^{m\times m}$ with poles on $\mathbb{T}$ and suppose that $A \subset H^p_m$ is the space of analytic functions in H^p_m that has an analytic extension to $\overline{\mathbb{D}}$, the closed unit disk. Then $A \subset \mathrm{Dom}\,(T_\Omega)$ and furthermore, A is a core of T_Ω.

Factorization

Classical Wiener-Hopf factorization

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Let $\Omega \in \mathrm{Rat}^{m\times m}$ with $\det \Omega \not\equiv 0$ and no poles or zeroes on $\mathbb{T}$. Then

$$\Omega = \Omega_{-} \mathrm{Diag}(z^{\kappa_j}) \Omega_{+}$$

where Ω_- and $(\Omega_-)^{-1}$ are minus functions , Ω_+ and $(\Omega_+)^{-1}$ are plus functions and $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m \in \mathbb{Z}$.

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Wiener-Hopf like factorization of rational matrix functions with poles and/or zeroes on $\mathbb{T}$

Let $\Omega \in \mathrm{Rat}^{m\times m}$ with $\det \Omega \not\equiv 0$. Then

$$\Omega = z^{-k}\Omega_-\Omega_0 P_0 \Omega_+, \text{ for some } k > 0,$$

Ω_- and $(\Omega_-)^{-1}$ are minus functions , Ω_+ and $(\Omega_+)^{-1}$ are plus functions, $\Omega_0 = \mathrm{Diag}_{j=1}^{m}(\phi_j)$ with ϕ_j a scalar rational function with poles and zeroes only on $\mathbb{T}$ and P_0 is a lower triangular polynomial matrix with $\det(P_0(z)) = z^N$ for some $N \ge 0$.

Factorization: example

$$\Omega = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}$$

Factorization: example

$$\Omega = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix} \text{ then}$$

$$\Omega = z^{-2}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$$

where

$$\Omega_-(z) = \begin{pmatrix} -\frac{1}{z} - 1 & -\frac{1}{z^2} \\ 1 & \frac{1}{z} - 1 \end{pmatrix}, \Omega_+(z) = \begin{pmatrix} -z & -1 \\ 1 & 0 \end{pmatrix},$$

$$\Omega_0(z) = \begin{pmatrix} z-1 & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} \quad \text{and } P_0(z) = \begin{pmatrix} 1 & 0 \\ z^3 + z^2 - z & z^4 \end{pmatrix}.$$

Factorization

Let $\Omega = z^{-k}\Omega_-\Omega_0 P_0 \Omega_+$ be the Wiener-Hopf type factorization of Ω as above. Put $\Xi = z^{-k}\Omega_0 P_0$ then

$$T_\Omega = T_{\Omega_-} T_\Xi T_{\Omega_+},$$

and

$$T_\Xi = T_{z^{-k} I_m} T_{\Omega_0} T_{P_0}$$

Fredholm: diagonal case

Suppose that $\Omega(z) \in \mathrm{Rat}^{m\times m}(\mathbb{T}\cup\{0\})$ with

$$\Omega(z) = \begin{pmatrix} \omega_1(z) & & & \\ & \omega_2(z) & & \\ & & \ddots & \\ & & & \omega_m(z) \end{pmatrix}$$

and $\omega_j(z) \in \mathrm{Rat}(\mathbb{T}\cup\{0\}), j = 1, 2, \ldots, m$. Then T_Ω is Fredholm if and only if T_{ω_j} is Fredholm for each $j = 1, \ldots m$, i.e., if and only if ω_j has no roots on $\mathbb{T}$ for each $j = 1, \ldots, m$. In case T_Ω is Fredholm we have

$$\text{Index } T_\Omega = \sum_{j=1}^{m} \text{Index } T_{\omega_j}.$$

Fredholm: nondiagonal case

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Let $\Omega(z) = z^{-k}\Omega_-(z)\Omega_0(z)P_0(z)\Omega_+(z)$ be the Wiener-Hopf type factorization of Ω as above. Put $\Xi(z) = z^{-k}\Omega_0(z)P_0(z)$. Then T_Ω is Fredholm if and only if T_Ξ is Fredholm if and only if T_{Ω_0} is Fredholm, which happens exactly when each of the entries ϕ_j of Ω_0 has no zeroes on $\mathbb{T}$. In case T_Ω is Fredholm,

$$\begin{aligned} \mathrm{Index}\ T_\Omega = \mathrm{Index}\ T_\Xi \quad &= mk + \mathrm{Index}\ T_{\Omega_0} + \mathrm{Index}\ T_{P_0} \\ &= mk + \sum_{j=1}^{m} \mathrm{degree}\, q_j - \sum_{j=1}^{m} k_j, \end{aligned}$$

where $\phi_j = \frac{s_j}{q_j}$ and q_j has roots only on $\mathbb{T}$ and k_j are the powers of z on the diagonal of P_0.

Fredholm: example

For $\Omega = \begin{pmatrix} 1 & \frac{1}{z-1} \\ 0 & 1 \end{pmatrix}$ then

$$\Omega(z) = z^{-2}\Omega_-(z) \begin{pmatrix} z-1 & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^3+z^2-z & z^4 \end{pmatrix} \Omega_+(z).$$

Since

$$\Omega_0(z) = \begin{pmatrix} z-1 & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix}$$

has a zero on $\mathbb{T}$ we have that T_Ω is not Fredholm even though $\det \Omega = 1$.

Fredholm: indices

Let

$$\Xi(z) = \Omega_0(z) P_0(z) = \begin{pmatrix} \frac{z^{k_1}}{q_1(z)} & 0 & \cdots & 0 \\ \frac{p_{2,1}}{q_2(z)} & \frac{z^{k_2}}{q_2(z)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{p_{m,1}}{q_m(z)} & \cdots & \frac{p_{m,m-1}}{q_m(z)} & \frac{z^{k_m}}{q_m(z)} \end{pmatrix}$$

Are the numbers $k_1, k_2, \ldots, k_m$ unique as in the classical (bounded) case, when Ξ results from a Wiener-Hopf factorization? Or, are the partial indices on the main diagonal

$\text{degree}\, q_j - k_j, \quad j = 1, \ldots, m$ unique?

Fredholm: indices

Suppose

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results from a Wiener-Hopf factorization with

$$k_j - \mathrm{degree}\, q_j \geq \mathrm{degree}\, p_{i,j} - \mathrm{degree}\, q_i,\, i = j+1 \ldots m.$$

If T_Ω is Fredholm then we can read the Fredholm partial indices of T_Ω from the main diagonal entries.

Fredholm: indices(example 1)

Let

$$\Xi(z) = \begin{pmatrix} \frac{1}{(z-1)^3} & 0 \\ \frac{z^2}{(z-1)^4} & \frac{z^5}{(z-1)^4} \end{pmatrix}$$

Then $\Xi(z)$ is in the required form and so $\mathrm{Index}\,(T_\Xi) = 3 - 1 = 2$. Note, though, that

$$\Xi(z) = \Xi_-(z)^{-1} \begin{pmatrix} \frac{z^3}{(z-1)^3} & 0 \\ 0 & \frac{z^2}{(z-1)^4} \end{pmatrix} \Xi_+(z)^{-1}$$

where

$$\Xi_-(z) = \begin{pmatrix} -1 & \frac{z-1}{z^2} \\ 0 & 1 \end{pmatrix}, \Xi_+(z) = \begin{pmatrix} -z^3 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that Ξ_- and its inverse are minus functions whereas Ξ_+ and its inverse are plus functions.

Fredholm: indices(example 2)

Let

$$\Xi_1(z) = \begin{pmatrix} \frac{1}{(z-1)^3} & 0 \\ \frac{z}{(z-1)^4} & \frac{z^5}{(z-1)^4} \end{pmatrix}$$

Then $\Xi_1(z)$ is in the required form and so again, $\mathrm{Index}\,(T_{\Xi_1}) = 3 - 1 = 2$. Note now, though, that

$$\Xi(z) = \Xi_-(z)^{-1} \begin{pmatrix} \frac{z^4}{(z-1)^3} & 0 \\ 0 & \frac{z}{(z-1)^4} \end{pmatrix} \Xi_+(z)^{-1}$$

where

$$\Xi_-(z) = \begin{pmatrix} -1 & \frac{z-1}{z} \\ 0 & 1 \end{pmatrix}, \Xi_+(z) = \begin{pmatrix} -z^4 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again Ξ_- and its inverse are minus functions whereas Ξ_+ and its inverse are plus functions.

The End

Thank you for your attention