Exact solutions of flow and pressure variation inside a horizontal filter chamber using Lie symmetry analysis.

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Abstract

The work focus on finding analytical and semi-analytical solutions of the momentum and pressure

variation inside a horizontal filter chamber.

The basic governing partial differential equations describing the flow and

pressure distribution will be

reduced to a system of ordinary differential equations using symmetry analysis.

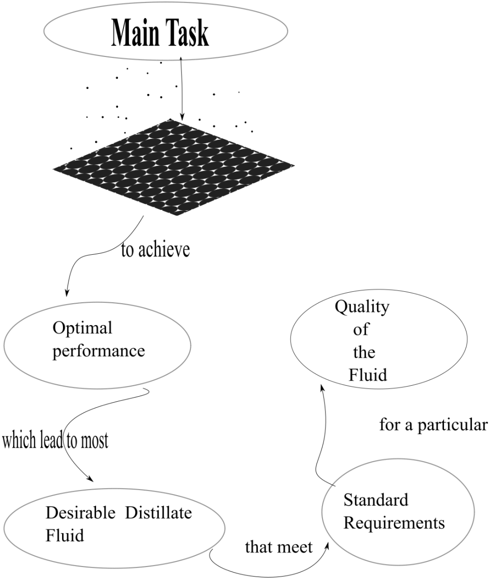
Finally, the effect of Stuarts number N, Permeation K and

Reynolds Number R_e momentum and pressure will be studied.

Introduction

- Advantages: $\mapsto$ Mathematical Modelling $\mapsto$ industrial case studies.
  - Understand complicated processes better
  - To increase profit
  - To improve or predict systems production.
- Industrial Fluids:$\not\Leftrightarrow$ components $\not\Leftrightarrow$ boiling points temperature, weights and sizes.
  - E.g Crude Oil (Found b\n Sedimentary rocks)
  - Ground Water
- **During manufacturing/fluid treatment process**:

Introduction



To interpret the experimental findings, designers need a theoretical base (models) for the design of practical and optimal filter systems.

Thus, mathematical modelling is a powerful tool that provides great insight of various physical phenomena.

In the past many theoretical attempts have been done on this topic.

Previous Work

- Analytical and Numerical Investigations
  - Berman [1]

    Steady flow, channel with stationary permeable walls, small R_e, perturbation.
  - Uchida and Aoki [2]

    Effects of wall expansion or contraction, understand unsteady blood flow.
  - Dauenhauer and Majdalani [3]

    Slowly deformable channel, permeable, unsteady, perturbation technique.
  - Majdalani *et al* [4]

    Application in transport of biological fluids, solution for small R_e, comparison with numerical solutions.
  - Domairry and Hatami [5]

    Nano-fluid flow and heat transfer between parallel plates, squeezing , DTM Padè method.

Motivation and Present Approach

- Previous research: Makes strong case for advantage of fluid injection in permeable channel.
- However, most of the studies did not investigate the effects of surface forces, body forces and heat transfer in different channel geometry (in our case: Horizontal).
- Most did not study the importance of specific application (In our case Filtration Process).
- Understanding flow, pressure distribution and analyse parameters affecting

  the flow and pressure distribution to find combination of parameters that lead

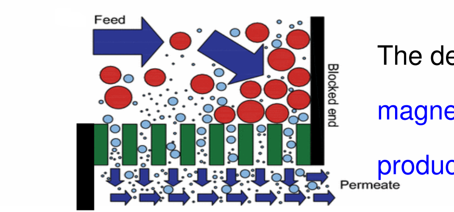
  to optimal filtration operation speed.

Problem statement

Several factors/parameters influence the design of a system. Due to this, the study of filtration process gained lot of attention over the years.

One of the most common challenges during filtration is flow restriction due to particles blocking the filter medium pores.

Problem statement



The design of the current study is such that magnet is integrated in the system to produce load zone that collect iron, steel particles, etc before fluid enters the chamber.

Thus, the permeable surface will filter less particles and pores will take time to become plugged with particles. Flow process speed is also a challenge.

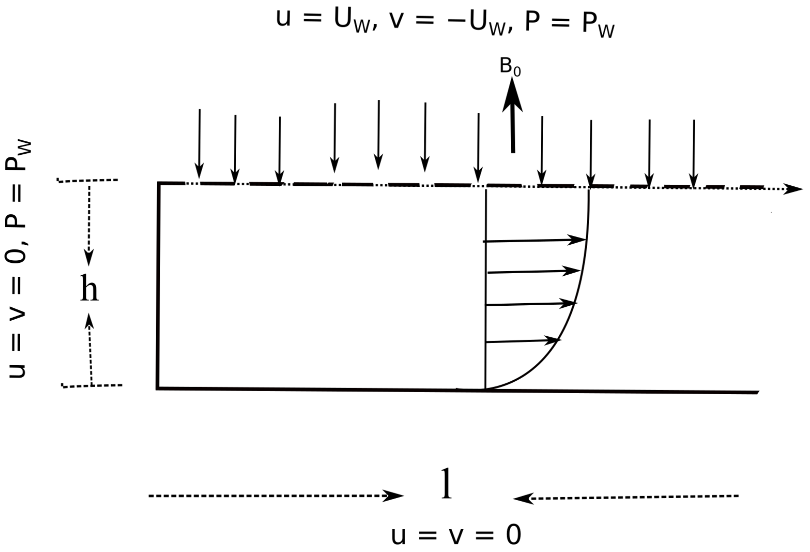
Hence, the current design is such that the filter chamber is horizontal.

Flow configurtion

The flow is treated as two-dimensional flow of an incompressible and electrically conducting viscous fluid in a horizontal porous rectangular channel bounded by permeable and impermeable walls kept at a constant temperature T_w.

Flow configuration

The physical model of the above specified flow is represented by the following



The mathematical model of the above specified flow is represented by conservation law of mass, momentum and energy respectively.

Governing equations and boundary conditions

$$\left.\begin{array}{r}\underbrace{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,}_{Continuity}\\[2ex] \underbrace{\frac{\partial u}{\partial t}}_{local}+\underbrace{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}}_{con.acc}=-\frac{1}{\rho}\underbrace{\frac{\partial P}{\partial x}}_{P.grad}+\nu\underbrace{\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]}_{Mom.diff}-\underbrace{\frac{\nu\phi}{k}u}_{flux\mapsto p.med}-\underbrace{\frac{\sigma B_0^2}{\rho}u}_{Lor.force},\\[2ex] \overbrace{\frac{\partial v}{\partial t}}+\overbrace{u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}}=-\frac{1}{\rho}\overbrace{\frac{\partial P}{\partial y}}+\nu\overbrace{\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]}-\overbrace{\frac{\nu\phi}{k}}v.\end{array}\right\} \quad (1)$$

where $u(t,x,y)$ is axial-velocity, $v(t,x,y)$ is normal velocity, ρ is density, $P(t,x,y)$ is

pressure, ν is kinetic viscosity, B_0 is the induced magnetic field, ϕ is the porosity

parameter and k is the permeability of the porous medium.
The appropriate boundary conditions according to the filter design are as follows:

$$
\begin{aligned}
&\text{(i)} \qquad u = U_w, \quad v = -U_w, \quad P = P_w, \quad \text{at } y = h,\\
&\text{(ii)} \qquad u = 0, \quad v = 0, \quad \text{at } y = 0.
\end{aligned}
\tag{2}
$$

Solution Method

This section aims to represent the dimensional system of equations (1) and boundary conditions (2) describing the problem at hand into the simpler

and solvable dimensionless representation of the case study using Lie symmetry analysis. Thereafter, we integrate the solvable system of equations together

with boundary conditions to obtain exact solutions of momentum and pressure variation inside the filter chamber.

Dimensional analysis

This subsection represents the system of equations describing the current case study in dimensionless form. The non-dimensional variables are

$$\bar{u} = \frac{u}{U_w}, \quad \bar{v} = \frac{v}{U_w}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{P} = \frac{P}{\rho U_w^2},$$
$$\bar{t} = \frac{tU_w}{L}, \quad N = \frac{\sigma h B_0^2}{\rho U_w}, \quad \frac{1}{K} = \frac{\nu \phi L}{k U_w}, \tag{3}$$

where $\bar{u}$ and $\bar{v}$ are the dimensionless velocity components, $\bar{P}$ is dimensionless pressure, $\bar{t}$ is dimensionless time, $\bar{x}$ and $\bar{y}$ are the dimensionless space coordinates,

L is the characteristic length, U_w is the upper wall velocity.
Substituting non-dimensional quantities (3) into a system of equations (1) and boundary equations (2)

and dropping the bars for simplicity yields the following non-dimensional representation of the current case study

Dimensional analysis

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{R_e}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]-\frac{1}{K}u-Nu,\\ &\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{R_e}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\frac{1}{K}v, \end{aligned}\right\} \tag{4}$$

Dimensional analysis

where $R_e = \dfrac{hU_w}{\nu}$ is the Reynolds number, $K = \dfrac{kU_w}{\nu\phi h}$ is the permeation parameter and $N = \frac{\sigma h B_0^2}{\rho U_w}$ is the Stuart number.
The boundary conditions take the following form

$$\begin{aligned} &\text{(i)} \qquad u = 1, \quad v = -1, \quad p = 1, \quad \text{at } y = \frac{h}{L} = H, \\ &\text{(ii)} \qquad u = 0, \quad v = 0, \quad \text{at } y = 0. \end{aligned} \tag{5}$$

According system model (4), there exists a non-dimensional stream velocity $\psi(t, x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{6}$$

which satisfies continuity equation from system (4) indentically. Thus, dimensionless system (4) becomes

$$\left.\begin{aligned} &\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} + P_x - \frac{1}{R_e}[\psi_{xxy} + \psi_{yyy}] + \frac{1}{K}\psi_y + N\psi_y = 0, \\ &\psi_{xt} + \psi_y\psi_{xx} - \psi_x\psi_{xy} + P_y - \frac{1}{R_e}[\psi_{xyy} + \psi_{xxx}] + \frac{1}{K}\psi_x = 0, \end{aligned}\right\} \tag{7}$$

Dimensional analysis

The dimensionless boundary conditions become

$$\begin{aligned} &\text{(i)} \qquad \psi = y - x + 1, \quad p = 1, \quad \text{at } y = \frac{h}{L} = H, \\ &\text{(ii)} \qquad \psi = 0, \quad \text{at } y = 0. \end{aligned} \tag{8}$$

The stream flow ψ above satisfies the internal positive flow stream according

to dimensionless model representing the current case study in a region $x, y > 0$ when $y - x + 1 > 0$ during operation when $t > 0$.

Symmetry reduction

Solving the resulting equations yields the following five Lie point symmetries spanned by system (7):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = F_1(t)\frac{\partial}{\partial \Psi}, \quad X_5 = F_2(t)\frac{\partial}{\partial P}. \tag{9}$$

For unsteady behaviour, we firstly consider the linear combination of X_1 and X_3, which gives $\frac{\partial}{\partial t} + c\frac{\partial}{\partial y}$, where c is the permeates travelling wave speed. The resulting characteristic equations give the following group invariants:

$$\psi = \psi(s,x), \quad p = p(s,x), \quad s = y - ct \quad \text{and} \quad x = x. \tag{10}$$

Substituting the above invariants (10) into system (7) yields

Symmetry reduction

$$\left.\begin{aligned} c\psi_{ss} + \frac{\psi_{xxs}}{R_e} + \frac{\psi_{sss}}{R_e} - \frac{\psi_s}{K} - N\psi_s - p_x - \psi_s\psi_{xs} + \psi_{ss}\psi_x = 0, \\ c\psi_{xs} + \frac{\psi_{xss}}{R_e} + \frac{\psi_{xxx}}{R_e} - \frac{\psi_x}{K} + p_s + \psi_x\psi_{xs} - \psi_s\psi_{xx} = 0, \end{aligned}\right\} \tag{11}$$

here ψ and P are stream velocity and pressure, respectively, while s and x

are new variables that affect filtration process dynamics.
Similarly, the above system (11) admits the following four Lie point symmetries:

$$X_1 = \frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial P}, \quad X_4 = \frac{\partial}{\partial \Psi}. \tag{12}$$

Thus, the linear combination of symmetries X_1 and X_2, $X = \frac{\partial}{\partial s} + \frac{\partial}{\partial x}$, yields the following invariants:

Symmetry reduction

$$\psi = f(g), \quad p = z(g) \quad \text{and} \quad g = x - s, \tag{13}$$

which further reduces system (7) into the following system of ODEs:

$$cf''(g) - \frac{2f'''(g)}{R_e} + \frac{f'(g)}{K} + Nf'(g) - z'(g) = 0, \tag{14}$$

$$cf''(g) - \frac{2f'''(g)}{R_e} + \frac{f'(g)}{K} + z'(g) = 0. \tag{15}$$

Here $f(g)$ is velocity, $z(g)$ is the pressure, and g is the combination of chamber space and time variables during the unsteady filtration process.

Closed-form Solutions

Thus, the axial and normal velocities and internal pressure are given by

$$
\begin{aligned}
u(t,x,y) &= \frac{e^{-\frac{(c-H+L)(A+2cKR_e)}{4K}}}{e^{\frac{HA}{2K}}-1}\left(e^{\frac{1}{4}\left[\frac{A(2c-ct+2L-x+y)}{K}+cR_e(g+c-H+L)\right]}\right.\\
&\left.-e^{\frac{1}{4}\left[\frac{g(A+cKR_e)}{K}+cR_e(c-H+L)\right]}\right),\\
v(t,x,y) &= \frac{e^{-\frac{(c-H+L)(A+2cKR_e)}{4K}}}{e^{\frac{gA}{2K}}-1}\left(e^{\frac{1}{4}\left[\frac{A(2c-ct+2L-x+y)}{K}+cR_e(g+c-H+L)\right]}\right.\\
&\left.-e^{\frac{1}{4}\left[\frac{g(A+cKR_e)}{K}+cR_e(c-H+L)\right]}\right),\\
p(t,x,y) &= \frac{2KN}{e^{\frac{HA}{2K}}-1}e^{\frac{g(cKR_e-A)}{4K}}\left(\frac{e^{\left[\frac{2gA-(c-H+L)(A+cKR_e)}{4K}\right]}}{A+cKR_e}+\frac{e^{\frac{1}{4}\left[\frac{(c+H+L)A}{K}-cR_e(c-H+L)\right]}}{A-cKR_e}\right)-\\
&\frac{NA\coth\left(\frac{HA}{4K}\right)}{2R_e(KN+2)}-\frac{(c-2)KN-4}{2(KN+2)},
\end{aligned}
$$

here $g = ct + x - y$ and $A = \sqrt{KR_e\left(c^2KR_e + 4KN + 8\right)}$.

Results and discussion

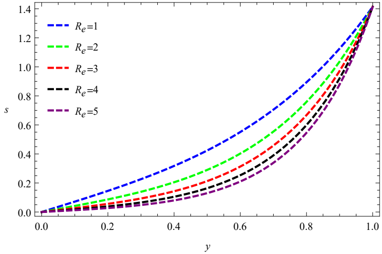


Figure: effects of Re on speed.

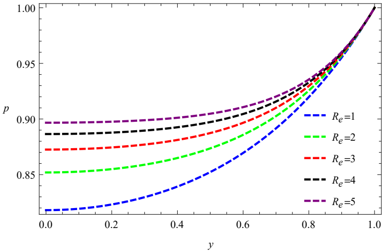


Figure: effects of Re on pressure.

Results and discussion

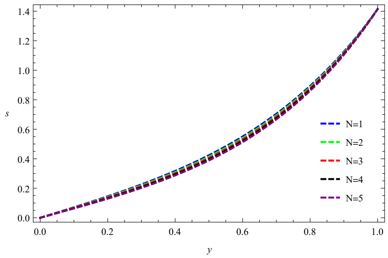


Figure: effects of stuart number on speed.

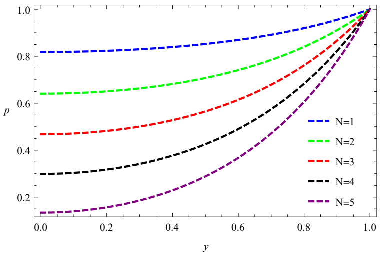


Figure: effects of stuart number on pressure.

Results and discussion

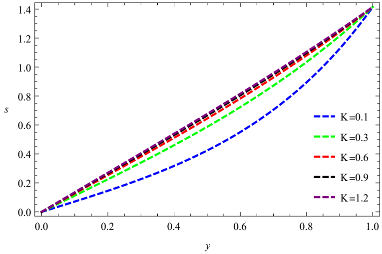


Figure: effects of permeation on speed.

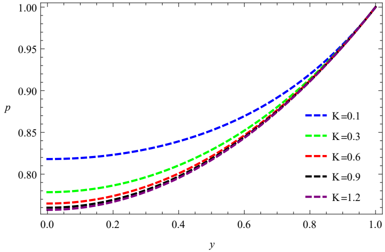


Figure: effects of permeation on pressure.

Conclusion

To achieve desirable permeates outflow while enhancing the system's internal pressure and minimising internal friction during unsteady-state filtration operation, the study led to the following conclusions:

Conclusion

i. To increase permeates outflow velocity, it is ideal to enhance internal pressure during operation to allow pressure pushing effects to drive fluid particles out of the chamber.

ii. During unsteady operation, it is ideal to increase the chamber volume since this allows fluid particles more space to move, thus increasing permeates outflow.

iii. When the operation is unsteady, for a porous medium to restrict contaminations while increasing permeate outflow velocity, it is ideal to decrease the pore size of the medium while injecting less fluid into the filter chamber to enable free movement of particles.

- A.S. Berman, Laminar flow in channels with porous walls, J. Appl. Phys. **24**, 1232-1235, 1953.
- S. Uchida and H. Aoki, "Unsteady flows in a semi-infinite contracting or expanding pipe," J. Fluid Mech, **82** (2) (1977), 371-387.
- E.C. Dauenhauer and J. Majdalani, Exact self similarity solution of the Navier-Stokes equations for a deformable channel with wall suction or injection, AIAA **3588**, 1-11, 2001.
- J. Majdalani, C. Zhou and C.A. Dawson, Two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, J. Biomech. **35**, 1399-1403, 2002.
- G. Domairry and M. Hatami, Squeezing Cu-water nanofluid flow analysis between parallel plates by DTM-Padé Method, Journal of Molecular Liquids, 193, (2014) 37-44.
- J. Lee, P. Kandaswamy, M. Bhuvaneswari and S. Sivasankaran, Lie group analysis of radiation natural convection heat transfer past an inclined porous surface, Journal of Mechanical Science and Technology, **22** (2008), 1779-1784.

- M. Mahmood, M.A. Hossain, S. Asghar and T. Hayat, Application of homotopy pertubation method to deformable channel with wall suction and injection in a porous medium, Int. J. Nonlinear Sci. Numerical Simulation **9**, 195-206, 2008.
- S. Asghar, M. Mushtaq and A.H. Kara, Exact solutions using symmetry methods and conservation laws for the viscous flow through expanding-contracting channels, Appl. Math. Modeling **32**, 2936-2940, 2008.
- Y.Z. Boutros, A.B. Abd-el-Malek, N.A. Badran and H.S. Hassan, Lie group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, Appl. Math. Modeling **31**, 1092-1108, 2007.
- D.Z. Noor, P.R. Kanna and M. Chern, Flow and heat transfer in a driven square cavity with double-sided oscillating lids in anti-phase, International Journal of heat and mass transfer **52**, 3009-3023, 2009.
- V.M. Mbugua, M. N. Kinyanjuni, K. Giterere, P. R. Kiogora, Hydromagnetic fluid flow between porous parallel plates in presence of varaible transverse magnetic field, Journal of Engineering and Innovative Technology, 4 (2015) 2248-2256.

- B. Lehnert, Dynamics of charged particles, 4th Edition, Proceeding of the Third international conference on the Physics of Electronic and Atomic Collisions, London.
- P. H. Roberts, An introduction to magnetohydrodynamics, 4th Edition, The Whitefriars Press Ltd., Great Britain.
- D. J. Griffiths, Introduction to electrodynamics, 3th Edition, Library of Congress Cataloging, United States of America.
- I. G. Currie, Fundamental Mechanics of fluids, 4th Edition, CRC Press, New York.

Thank you!