STAR MULTIPLICATION AND CROSSED MODULES STRUCTURES IN RIGHT Ω-LOOPS

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Contents

Abstract

Internal categorical structures play an important role in categorical algebra. In semi-abelian categories, internal categories form a variety, namely the variety of crossed modules [10]. The notion of star multiplication was introduced by G. Janelidze in [10] where it was applied to the description of crossed modules in a semi-abelian category. In the same paper, the question of describing semi-abelian categories with the property that every star-multiplicative graph uniquely extends to an internal category structure was asked.
N. Martins-Ferreira [14] introduced conditions that provide a simple description of internal groupoids as crossed modules in the semi-abelian categories of groups and rings. The aim of this talk is to describe star-multiplication in varieties of right Ω-loops in the sense of [9].

Outline

- Notation and preliminaries
- Internal actions and semidirect products in Ω-loops
- Internal reflexive graphs, categories and groupoids
- Star multiplication
- References

Let **C** be a finitely complete category with coproducts.
$Pt_{\mathbf{C}}(B)$, the category of points over B:
The objects in $Pt_{\mathbf{C}}(B)$ are triples (A, α, β), where $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow A$ are morphisms in **C** in with $\alpha\beta = 1_B$. A morphism $f : (A, \alpha, \beta) \longrightarrow (A', \alpha', \beta')$ in $Pt_{\mathbf{C}}(B)$ is a morphism $f : A \longrightarrow A'$ in **C** with $\alpha' f = \alpha$ and $f\beta = \beta'$.

Protomodular and semiabelian categories provide the appropriate setting for describing the essential features of groups and other algebraic structures. A category **C** is semi-abelian (Janelidze, Marki and Tholen, [11]) when

(a) **C** is pointed.

(b) **C** is finitely complete.

(c) **C** is finitely cocomplete.

(d) **C** is Barr-exact (M. Barr, 1971).
  - regular: finitely complete with pullback stable regular epis and coequalizers of kernel pairs
  - every internal equivalence relation is a kernel pair

(e) **C** is protomodular (Bourn, 1991); Pullback functor $f^* : Pt(B) \longrightarrow Pt(E)$ conservative $\Longleftrightarrow$ short 5 lemma (in pointed case)

Let **C** be a finitely complete category with coproducts. Then for any object B in **C**, we can define a functor U from the category of split epimorphisms over B into **C**

$$U : Pt_B(\mathbf{C}) \longrightarrow \mathbf{C},\ A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longmapsto ker(\alpha).$$

This functor has a left adjoint $F : \mathbf{C} \longrightarrow Pt_B(\mathbf{C}),\ X \longmapsto B + X \underset{\iota_B}{\overset{[1,0]}{\rightleftarrows}} B$

and the monad corresponding to this adjunction is denoted by $B\flat(-)$. The $B\flat(-)$-algebras are called B-actions in **C** (See [4]).

T is the monad on $\mathbf{C}$ determined by the adjoint pair (F, U), $C^T = B\flat X$ the category of T-algebras, and U^T, F^T, and K are the corresponding forgetful functor, free functor, and comparison functor respectively. L is the left adjoint of K.

Remarks 1

- *We write*

$$B\flat X \xrightarrow{\kappa_{B,X}} B+X \xrightarrow{[1,0]} B \qquad (1)$$

for the diagram in which $+$ *denotes the coproduct and* $\kappa_{B,X}$ *is the kernel of the morphism* $[1,0]$ *induced by the identity morphism of* B *and the zero morphism from* B *to* X.

- *In the category of groups, the object* $B\flat X$ *is the group generated by the formal conjugates of elements of* B *by the triples* (b, x, b^{-1}) *with* $b \in B$ *and* $x \in X$.

Definition 2

The algebras for the monad $B\flat(-)$ (or T^B) induced by the adjunction (F, U) called internal B-actions in $\mathbf{C}$. We denote by $\mathbf{C}^B$ the category of these algebras.

Definition 3

(D. Bourn and G. Janelidze, [1])
Let T^B *be the monad on* $\mathbb{C}$ *corresponding to the monadic functor* $U : Pt_{\mathbb{C}}(B) \to \mathbb{C}$*. Given a* T^B*-algebra* (X, ξ)*, we define the semidirect product*

$$B \ltimes (X, \xi)$$

to be the object in $Pt_{\mathbb{C}}(B)$ *corresponding to* (X, ξ) *under the category equivalence*

$$Pt_{\mathbb{C}}(B) \approxeq \mathbb{C}^{T^B}$$

. *i.e*

$$L(X, \xi) = (B \ltimes (X, \xi), \pi_\xi, \iota_\xi)$$

Remarks 4

- $B \ltimes (X,\xi) = L(X,\xi)$ *is defined via the coequalizer diagram*

$$B\flat X \underset{\xi}{\overset{\kappa_{B,X}}{\rightrightarrows}} B + X \xrightarrow{q_\xi} L(X,\xi) \qquad (2)$$

- *For* (A,α,β) *in* $Pt_{\mathsf{C}}(B)$, *consider the diagram*

$$\begin{array}{ccccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X & \underset{\iota_1}{\overset{[1,0]}{\rightleftarrows}} & B \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle [\beta,\kappa]} & & \| \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array} \qquad (3)$$

where $(B\flat X, \kappa_{B,X})$ *is the kernel of* $[1,0]$, (X,κ) *is the kernel of* α, *and* ξ *is the induced morphism between these kernels. We can write* $K(A,\alpha,\beta) = (X,\xi)$.

Theorem 5 ([9])

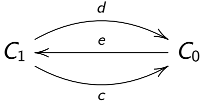
In Ω-**Loop** *Given an object* B *and a* T^B*-algebra* (X, ξ)*, the semidirect product* $B \ltimes (X, \xi)$ *is the set-theoretical (cartesian) product* $B \times X$ *equipped with the following* Ω*-algebra structure:*

$$\omega((b_1, x_1), ..., (b_n, x_n)) = (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \tag{4}$$

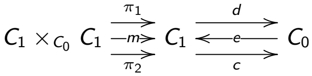
for each n-ary operation $\omega \in \Omega$ *and for all* $b_1, ..., b_n \in B, \quad x_1, ..., x_n \in X$.

Internal reflexive graphs, internal categories and crossed modules

Let **C** be a category with pullbacks. Recall that an internal reflexive graph in C is give by a diagram



where C_0 is called the "object of objects" and C_1, the "object of arrows", such that the domain morphism d, the codomain morphism c and the identity morphism e satisfy $de = de = 1_{C_0}$.
C is endowed with an internal category structure



when there is a multiplication $m : C_1 \times_{C_0} C_1 \longrightarrow C_1$ satisfying the axioms

C1 $m(1_{d(x)}, x) = x = m(x, 1_{d(x)})$

C2 $d(m(x, y)) = d(x)$

C3 $c(m(x, y)) = c(y)$

C4 $m(x, m(y, z)) = m(m(x, y), z)$

where $1_X = e(X)$ and $C_1 \times_{C_0} C_1$ denotes the pullback of d and c.

We shall denote by $cat(\mathbf{C})$ the category of internal categories in C and by $RG(\mathbf{C})$ that of internal reflexive graphs.
In [6] it was shown that the notions of reflexive multiplicative graph, internal category and internal groupoid coincide

Theorem 6

If C is a C is a Maltsev category, then

1. any internal reflexive graph admits at most one structure reflexive-multiplicative graph;
2. $V: \ Cat(C) \hookrightarrow RMG(\mathbf{C})$ and $U: \ Grpd(C) \hookrightarrow Cat(\mathbf{C})$ are isomorphisms.

There are several algebraic and combinatorial categories equivalent to the category of crossed modules (see Brown and Spencer [5], Porter [17], for a more categorical setting). A paper for this kind of result in a semiabelian setting is [10].
One of the categories equivalent to crossed modules of groups is Cat^1-**Groups**, the category of Cat^1-groups.

Recall:

Definition 7

A crossed module (A, B, δ) *consists of a homomorphism* $\delta : A \to B$ *together with an action of the group* B *on* A*, written as* $b.a$ *such that*

(a) $\delta(a).a' = a + a' - a$

(b) $\delta(b.a) = b + \delta(a) - b$

for all $a, a' \in A$ *and* $b \in B$*.*

In a semi-abelian category internal crossed modules are equivalent to internal categories [10].

Definition 8

An internal precrossed module in a semi-abelian category $\mathbf{C}$ *is a 4-tuple* (B, X, ξ, δ) *in which* (X, ξ) *is a* B*-action and* $\delta : X \longrightarrow B$ *a morphism in* $\mathbf{C}$ *such that the diagram*

$$\begin{array}{ccc} B \flat X & \xrightarrow{\kappa_{B,X}} & B + X \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle [1,\delta]} \\ X & \xrightarrow{\ \ \delta\ \ } & B \end{array} \tag{5}$$

commutes.

Definition 9

An internal crossed module in a semi-abelian category $\mathbf{C}$ *is an internal precrossed module* (B, X, ξ, δ) *for which the diagram*

$$\begin{array}{ccc} (B+X)\flat X & \xrightarrow{[1_B,\delta]\flat 1_X} & B\flat X \\ {\scriptstyle [1_{B+X},\iota_2]^\sharp}\downarrow & & \downarrow{\scriptstyle \xi} \\ B\flat X & \xrightarrow[\xi]{} & X \end{array} \tag{6}$$

commutes. Here, $[1_{B+X}, \iota_2]^\sharp$ *is the unique morphism such that* $\kappa_{B,X}[1_{B+X}, \iota_2]^\sharp = [1_{B+X}, \iota_2]\kappa_{B+X,X}$.

Theorem 10

A crossed module in a variety $\mathbf{V}$ *of right* Ω*-loops is a quadruple* (B, X, ξ, δ) *in which* (X, ξ) *is a* B*-action and* $\delta : X \longrightarrow B$ *is a morphism such that for an* n*-ary operation* $\omega \in \Omega$,

(i)

$$\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n) - \omega(b_1, ..., b_n)$$
$$= \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))$$

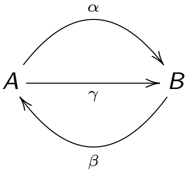
(ii)

$$\xi(\omega(x_1' + (\delta(x_1) + b_1), ..., x_n' + (\delta(x_n) + b_n)) -$$
$$\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n)) +$$
$$\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))$$
$$= \xi(\omega((x_1' + x_1) + b_1, ..., (x_n' + x_n) + b_n) - \omega(b_1, ..., b_n))$$

Star multiplication

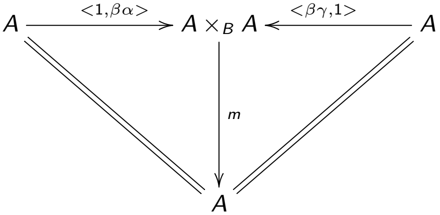
Definition 11

A multiplicative graph is a reflexive graph



(7)

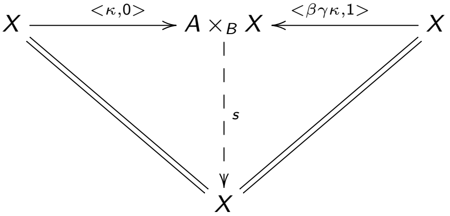
together with a binary multiplication, that is, a morphism $A \times_B A \xrightarrow{m} B$ *such that the following diagram commutes;*



(8)

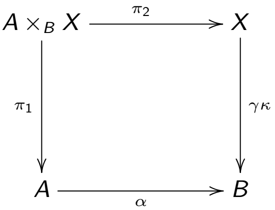
Definition 12

[10] Let $S = (A, B, \alpha, \beta, \gamma)$ *be an internal reflexive graph in a pointed category* $\mathbf{C}$ *with finite limits, and* $\kappa : X \longrightarrow A$ *a (fixed) kernel of* α*. The graph* S *is said to be star-multiplicative if there exists a unique morphism* $s : A \times_B X \longrightarrow X$ *making the diagram*



(9)

commute; here $A \times_B X$ *is defined as the pullback*



(10)

Can every star-multiplicative graph in a semiabelian category be equipped with a unique internal groupoid structure? [10]

1. Conditions that provide a description of internal groupoids as crossed modules in the semi-abelian categories of groups and rings provided in [14].
2. Using commutator theory, N. Martins-Ferreira and T. Van der Linden proved that in a semiabelian category, a reflexive graph has the "Smith is Huq" or (SH) property if and only if every star-multiplicative graph is a groupoid [15].
   i.e If the Smith commutator coincides with the Huq commutator, then every star-multiplicative graph uniquely extends to an internal category structure.

Examples of semi-belian categories that satisfy the Smith is Huq Condition (SH) are:

- groups
- rings, not necessarily unitary
- all Orzech categories of interest [16]

The category of loops is semi-abelian but does not satisfy SH ([7], Example 4.9)
G.Janelidze, L. Marki and S. Veldsman [12]
In the semi-abelian category of nearrings,it was shown by an counter-example, that the Huq and Smith commutators of ideals (normal subobjects) need not coincide.

Remark 13

A near-ring is a system $N = (N, 0, +, -, \cdot)$ *in which* $(N, +)$ *is a group (not necessarily abelian),* $(N, \cdot)$ *is a semigroup, and the right distributive law* $(x + y) \cdot z = a \cdot z + y \cdot z$ *holds.*

Theorem 14

*An internal crossed module (B, X, ξ, δ) in Ω-**RLoop** corresponds to a star-multiplicative graph under the equivalence*

$$PXMod(\Omega\text{-}\mathbf{RLoop}) \approx RG(\Omega\text{-}\mathbf{RLoop}) \tag{11}$$

if and only if

$$\begin{aligned}\xi(\omega(\delta(x_1') + x_1, ..., \delta(x_n') + x_n) - \omega(x_1, ..., x_n)) \\ = \omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n).\end{aligned} \tag{12}$$

for all $x_1, ..., x_n \in X,\ x_1', ..., x_n' \in X$.

Proof

Consider a diagram in Ω-**RLoop** of the form

$$X \xrightarrow{<\kappa,0>} A \times_B X \underset{<\beta\gamma\kappa,0>}{\overset{\pi_2}{\rightleftarrows}} X \,. \tag{13}$$

If $\bar{\xi} : X \flat X \longrightarrow X$ is the X-action on X corresponding to this split epimorphism, then $\bar{\xi}$ making the diagram

$$\begin{array}{ccc} X \flat X & \xrightarrow{\kappa_{X,X}} & X + X \\ {\scriptstyle \bar{\xi}} \downarrow & & \downarrow {\scriptstyle [<\beta\gamma\kappa,1>,<\kappa,0>]} \\ X & \xrightarrow[<\kappa,0>]{} & A \times_B X \end{array} \tag{14}$$

commute; that is to say that

$$< \kappa, 0 > \bar{\xi} = [< \beta\gamma\kappa, 1 >, < \kappa, 0 >]\kappa_{X,X} \tag{15}$$

Proof

This is equivalent to the requirement that the diagram

$$
\begin{array}{ccc}
X \flat X & \xrightarrow{\kappa_{X,X}} & X + X \\
\Big\downarrow{\scriptstyle \bar{\xi}} & & \Big\downarrow{\scriptstyle [<\beta\gamma\kappa,\kappa>]} \\
X & \xrightarrow[\kappa]{} & A
\end{array}
\tag{16}
$$

commutes.

Proof

By chasing this diagram, we get

$$\begin{aligned} \kappa\bar{\xi} = [\beta\gamma\kappa, \kappa]\kappa_{X,X} \\ \iff \kappa\bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ = [\beta\gamma\kappa, \kappa]\kappa_{X,X}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ \iff \bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ = [\beta\gamma\kappa, \kappa](\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ \iff \bar{\xi}(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ = \omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n) \end{aligned} \tag{17}$$

Proof

An internal crossed module (B, X, ξ, δ) will then correspond to a star-multiplicative graph under the equivalence

$$PXMod(\Omega\text{-}\mathbf{RLoop}) \approx RG(\Omega\text{-}\mathbf{RLoop}) \tag{18}$$

if and only if the diagram

$$\begin{array}{ccc} X \flat X & \xrightarrow{\bar{\xi}} & X \\ {\scriptstyle \delta \flat 1}\downarrow & & \parallel \\ B \flat X & \xrightarrow[\xi]{} & X \end{array} \tag{19}$$

commutes.

Proof

By the commutativity of this diagram, we have

$$\begin{aligned} \xi(\delta\flat 1)(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ = \xi(\omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n)) \\ \Longleftrightarrow \xi(\omega(\delta(x_1') + x_1, ..., \delta(x_n') + x_n) - \omega(x_1, ..., x_n)) \\ = \omega(x_1' + x_1, ..., x_n' + x_n) - \omega(x_1, ..., x_n) \end{aligned} \tag{20}$$

Corollary 15

In the category **Grp** *of groups every star-multiplicative graph extends to an internal category structure.*

Proof.
In equation 14 let ω be the binary $+$. Then

$$\begin{aligned} &\xi((\delta(x_1') + x_1) + (\delta(x_2') + x_2) - (x_1 + x_2)) \\ &\qquad = (x_1' + x_1) + (x_2' + x_2) - (x_1 + x_2) \\ &\iff \xi(\delta(x_1') + x_1 + \delta(x_2') - x_1) = x_1' + x_1 + x_2' - x_1 \end{aligned} \tag{21}$$

Since this equation holds for all $x_1, x_1', x_2, x_2' \in X$, we can substitute $x_1' = 0$. We then have

$$\xi(x_1 + \delta(x_2') - x_1) = x_1 + x_2' - x_1; \tag{22}$$

that is, $\delta(x_1).x_2' = x_1 + x_2' - x_1$, (the Peiffer condition for crossed modules in groups).
This confirms (see [10], Remark 4.7) that every star multiplicative graph in the category of groups is an internal category. □

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