Split extension cores for internal semi-abelian algebras in a cartesian closed category

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Actions and split extensions

- In the category of groups each split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \qquad \kappa = \ker(\alpha) \qquad \alpha\beta = 1_B$$

  of B with kernel X, determines an action of B on X, that is, a map $\cdot : B \times X \to X$ satisfying:
  1. $e \cdot x = x$ (where e is the identity element in B);
  2. $b \cdot (b' \cdot x) = (bb') \cdot x$;
  3. $b \cdot (xx') = (b \cdot x)(b \cdot x')$.

  This action can be defined by $\kappa(b \cdot x) = \beta(b)\kappa(x)\beta(b)^1$.
- A split extension can be recovered (up to isomorphism) from an action via a semi-direct product and every action action arises in this way. (In fact there is an equivalence of categories between the category of split extensions of B and the category of B-groups, i.e. groups equipped with an action of B on them).

Action cores

- Given a group B, a B-group X and a subgroup S of X, the action core of S with respect to X is the largest sub-B-group of X contained in X. The action core always exists and has underlying set
$$\{x \in X \mid \forall b \in B, b \cdot x \in S\}.$$
- All of the above can be generalized to *semi-abelian* categories, in the sense of Janelidze, Marki, Tholen [JMT02], by replacing group action by *internal object action* first studied by Bourn and Janelidze in [BJ98] whose name derives from [BJK05] of Borceux, Janelidze and Kelly. This produces the notion action core in sense of [BCGV22] which unlike in the case of the category of groups do not necessarily exist in every semi-abelian category. We won't go into the details of this here, instead we will work with split extensions.

Split extensions cores

- Given a split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{1}$$

and a monomorphism $m : S \to X$ in a pointed finitely complete category $\mathbb{C}$, the split extension core of S with respect (1) [BCGV22] is the terminal object in the category of morphisms of split extensions of the form

$$\begin{array}{ccccc} X' & \xrightarrow{\kappa'} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B \\ {\scriptstyle u}\downarrow & & \Big\downarrow & & \Big\| \\ S & & {\scriptstyle v} & & \\ {\scriptstyle m}\downarrow & & \Big\downarrow & & \Big\| \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array} \tag{2}$$

Split extension cores

- We will denote the split extension core as follows:

$$\begin{array}{ccccc} \bar{X} & \xrightarrow{\bar{\kappa}} & \bar{A} & \underset{\bar{\beta}}{\overset{\bar{\alpha}}{\rightleftarrows}} & B \\ {\scriptstyle \bar{u}}\downarrow & & & & \| \\ S & & \downarrow{\scriptstyle \bar{v}} & & \| \\ {\scriptstyle m}\downarrow & & & & \| \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B. \end{array} \qquad (3)$$

  It turns out that whenever the split extension core exists (in the finitely complete context) that $\bar{u}$ and $\bar{v}$ are necessarily monomorphisms.

- In groups $\bar{X} = \{x \in X \mid \forall b \in B, \beta(b)\kappa(x)\beta(b)^{-1} \in m(S)\}$ and $m\bar{u}$ is the inclusion of $\bar{X}$ in X (we do not describe the remainder of (4) since it is determined up to isomorphism).

A construction for split extension cores in semi-abelian varieties

- A variety of universal algebras $\mathcal{V}$ is semi-abelian if it has a unique constant e and admits, for some natural number n, an $n+1$-ary term p and n binary terms s_1, .. ,s_n satisfying: $s_i(x,x) = e$ and $p(s_1(x,y), ..., s_n(x,y), y) = x$.
- Examples include (Ω-) groups where $n = 1$, $p(x,y) = xy$ and $s_1(x,y) = xy^{-1}$, which include not necessarily unital rings and Lie algebras amongst others as well as, Heyting semi-lattices where $n = 2$, $p(x,y,z) = (x \to z) \wedge y$, $s_1(x,y) = x \to y$ and $s_2(x,y) = ((x \to y) \to y) \to x$ (Johnstone [Joh04]).

- For a natural number n we call an $n+1$-ary terms t a unary ideal term if $t(e, v_1, .., v_n) = e$ (which is a special case of ideal term in the sense of Ursini [Urs72]). We denote the set of all $n+1$-ary ideal terms I_n. For example for the variety of groups the term $t(x, y) = xyx^1$ is a unary ideal term.
- Given a split extension (1) and an element x in X the sub split extension of (1) with codomain B generated by x has kernel $\langle x \rangle = \{t(x, \beta(b_1), ..., \beta(b_n)) \mid n \in \mathbb{N}, t \in I_n, b_1, ..., b_n \in B\}$ i.e forms part of the smallest split extension

$$\begin{array}{ccccc} \langle x \rangle & \longrightarrow & \bullet & \rightleftarrows & B \\ \downarrow & & \downarrow & & \| \\ X & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B. \end{array} \quad (4)$$

with kernel containing x.

Proposition

$\mathcal{V}$ admits split extension cores if and only if for each split extension (1) and each sub-algebra S of X the set $\{x \mid \langle x \rangle \subseteq S\}$ forms a sub-algebra of X. In this case $\bar{X}$ is given by this sub-algebra.

Cartesian closed categories

- A small complete category $\mathbb{C}$ is cartesian closed if for each object B the functor $B \times - : \mathbb{C} \to \mathbb{C}$ has a right adjoint. Equivalently, for each C in $\mathbb{C}$ there exists an object C^B together with a morphism $\mathrm{ev} : C^B \times B \to C$ with the following universal property: for each object A together with a morphism $h : A \times B \to C$ there exists a unique morphism $\tilde{h} : A \to C^B$ such that $\mathrm{ev}(\tilde{h} \times 1) = h$.
- The category of sets is an example of such a category where $C^A = \{f \mid f \text{ is a function from } A \text{ to } C\}$ and $\mathrm{ev}(f)(a) = f(a)$.

Internal algebras

- An internal group (G, e, m, i) in a finitely complete category $\mathbb{C}$ consists of an object G in $\mathbb{C}$, and morphisms $e : 1 \to G$, $m : G \times G \to G$ and $i : G \to G$ (where 1 is the terminal object in $\mathbb{C}$) such that the following diagrams

$$\begin{array}{ccccc} G \times (G \times G) & & \xrightarrow{\alpha} & & (G \times G) \times G \\ {\scriptstyle 1\times m}\downarrow & & & & \downarrow{\scriptstyle m\times 1} \\ G \times G & \xrightarrow[m]{} & G & \xleftarrow[m]{} & G \times G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\langle 1,e!\rangle} & G \times G & \xleftarrow{\langle e!,1\rangle} & G \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1} & \\ & & G & & \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\langle 1,i\rangle} & G \times G & \xleftarrow{\langle i,1\rangle} & G \\ & {\scriptstyle e!}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle e!} & \\ & & G & & \end{array}$$

  commute.
- Other internal algebras can be suitably defined by replacing the underlying set by an object, the operations by morphisms, and axioms by commutative diagrams.

Split extension cores for internal semi-abelian algebras

- Given a split extension (1) and a monomorphism $m : S \to X$ we construct the object $\bar{X}$ (essentially copying what happens for sets) as the meet (wide pullback) of the family of monomorphisms $u_t : X_t \to X$, indexed by unary ideal terms t, obtained via pullback

$$
\begin{array}{ccc}
X_t & \xrightarrow{u_t} & X \\
{\scriptstyle p_t}\downarrow & & \downarrow{\scriptstyle \tilde{t}} \\
S^{B^n} & \xrightarrow[m^{B^n}]{} & X^{B^n}
\end{array}
$$

where $\tilde{t}$ is the unique morphism making the diagram

$$
\begin{array}{ccc}
X \times B^n & \xrightarrow{\tilde{t}\times 1} & X^{B^n} \times B^n \\
 & \searrow^{\underline{t}} & \downarrow{\scriptstyle \mathrm{ev}} \\
{\scriptstyle \kappa\times\beta^n}\downarrow & & X \\
 & & \downarrow{\scriptstyle \kappa} \\
A \times A^n & \xrightarrow[t]{} & A
\end{array}
$$

commute.

Split extension cores for internal semi-abelian algebras

Theorem

Let $\mathcal{V}$ be a variety of semi-abelian algebras and let $\mathbb{C}$ be a small complete cartesian closed category with small subobject lattice. If $\mathcal{V}$ admits split extension cores, then the category of internal $\mathcal{V}$ algebras in $\mathbb{C}$ admits split extension cores.

Idea of the proof

- Show that construction given for $\bar{X}$ works in the case where $\mathbb{C} = \mathbf{Set}^{\mathbb{X}}$ for a category $\mathbb{X}$.
- Use the fact that the yoneda embeding $Y : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{op}}$ preserves limits exponents and is fully faithful to show that $\overline{Y(X)} = Y(\bar{X})$ and use this to prove that $\bar{X}$ is a sub-algebra of X.

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