# Split extension cores for internal semi-abelian algebras in a cartesian closed category

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#### Actions and split extensions

In the category of groups each split extension

$$X \xrightarrow{\kappa} A \xrightarrow{\alpha} B$$
  $\kappa = \ker(\alpha)$   $\alpha\beta = 1_B$ 

of B with kernel X, determines an action of B on X, that is, a map  $\cdot : B \times X \to X$  satisfying:

- 1.  $e \cdot x = x$  (where e is the identity element in B);
- 2.  $b \cdot (b' \cdot x) = (bb') \cdot x$ ;
- 3.  $b \cdot (xx') = (b \cdot x)(b \cdot x')$ .

This action can be defined by  $\kappa(b \cdot x) = \beta(b)\kappa(x)\beta(b)^{1}$ .

A split extension can be recovered (up to isomorphism) from an action via a semi-direct product and every action action arises in this way. (In fact there is an equivalence of categories between the category of split extensions of B and the category of B-groups, i.e. groups equipped with an action of B on them).

#### Action cores

▶ Given a group B, a B-group X and a subgroup S of X, the action core of S with respect to X is the largest sub-B-group of X contained in X. The action core always exists and has underlying set

$$\{x \in X \mid \forall b \in B, b \cdot x \in S\}.$$

▶ All of the above can be generalized to *semi-abelian* categories, in the sense of Janelidze, Marki, Tholen [JMT02], by replacing group action by *internal object action* first studied by Bourn and Janelidze in [BJ98] whose name derives from [BJK05] of Borceux, Janelidze and Kelly. This produces the notion action core in sense of [BCGV22] which unlike in the case of the category of groups do not necessarily exist in every semi-abelian category. We won't go into the details of this here, instead we will work with split extensions.

#### Split extensions cores

► Given a split extension

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} B \tag{1}$$

and a monomorphism  $m: S \to X$  in a pointed finitely complete category  $\mathbb{C}$ , the split extension core of S with respect (1) [BCGV22] is the terminal object in the category of morphisms of split extensions of the form

$$X' \xrightarrow{\kappa'} A' \xrightarrow{\alpha'} B$$

$$X' \xrightarrow{\kappa'} A' \xrightarrow{\beta'} B$$

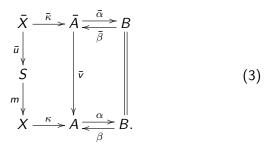
$$X \xrightarrow{\kappa} A \xrightarrow{\alpha} B$$

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#### Split extension cores

We will denote the split extension core as follows:



It turns out that whenever the split extension core exists (in the finitely complete context) that  $\bar{u}$  and  $\bar{v}$  are necessarily monomorphisms.

▶ In groups  $\bar{X} = \{x \in X \mid \forall b \in B, \beta(b)\kappa(x)\beta(b)^{-1} \in m(S)\}$  and  $m\bar{u}$  is the inclusion of  $\bar{X}$  in X (we do not describe the remainder of (4) since it is determined up to isomorphism).

## A construction for split extension cores in semi-abelian varieties

- A variety of universal algebras  $\mathcal{V}$  is semi-abelian if it has a unique constant e and admits, for some natural number n, an n+1-ary term p and n binary terms  $s_1, ..., s_n$  satisfying:  $s_i(x,x)=e$  and  $p(s_1(x,y),...,s_n(x,y),y)=x$ .
- Examples include (Ω-) groups where n=1, p(x,y)=xy and  $s_1(x,y)=xy^{-1}$ , which include not necessarily unital rings and Lie algebras amongst others as well as, Heyting semi-lattices where n=2,  $p(x,y,z)=(x\to z)\land y$ ,  $s_1(x,y)=x\to y$  and  $s_2(x,y)=((x\to y)\to y)\to x$  (Johnstone [Joh04]).

- For a natural number n we call an n+1-ary terms t a unary ideal term if  $t(e, v_1, ..., v_n) = e$  (which is a special case of ideal term in the sense of Ursini [Urs72]). We denote the set of all n+1-ary ideal terms  $I_n$ . For example for the variety of groups the term  $t(x, y) = xyx^1$  is a unary ideal term.
- ▶ Given a split extension (1) and an element x in X the sub split extension of (1) with codomain B generated by x has kernel  $\langle x \rangle = \{t(x, \beta(b_1), ..., \beta(b_n)) \mid n \in \mathbb{N}, t \in I_n, b_1, ..., b_n \in B\}$  i.e forms part of the smallest split extension

$$\begin{array}{cccc}
\langle x \rangle & \longrightarrow \bullet & \longrightarrow B \\
\downarrow & & \parallel & \\
X & \stackrel{\kappa}{\longrightarrow} A & \stackrel{\alpha}{\longrightarrow} B.
\end{array}$$
(4)

with kernel containing x.

#### Proposition

 $\mathcal V$  admits split extension cores if and only if for each split extension (1) and each sub-algebra S of X the set  $\{x \mid \langle x \rangle \subseteq S\}$  forms a sub-algebra of X. In this case  $\bar{X}$  is given by this sub-algebra.

#### Cartesian closed categories

- A small complete category  $\mathbb C$  is cartesian closed if for each object B the functor  $B \times : \mathbb C \to \mathbb C$  has a right adjoint. Equivalently, for each C in  $\mathbb C$  there exists an object  $C^B$  together with a morphism  $\mathrm{ev}: C^B \times B \to C$  with the following universal property: for each object A together with a morphism  $h: A \times B \to C$  there exists a unique morphism  $\tilde{h}: A \to C^B$  such that  $\mathrm{ev}(\tilde{h} \times 1) = h$ .
- The category of sets is an example of such a category where  $C^A = \{f \mid f \text{ is a function from } A \text{ to } C\}$  and ev(f)(a) = f(a).

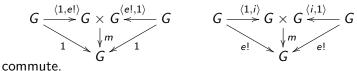
#### Internal algebras

An internal group (G, e, m, i) in a finitely complete category  $\mathbb C$  consists of an object G in  $\mathbb C$ , and morphisms  $e: 1 \to G$ ,  $m: G \times G \to G$  and  $i: G \to G$  (where 1 is the terminal object in  $\mathbb C$ ) such that the following diagrams

$$G \times (G \times G) \xrightarrow{\alpha} (G \times G) \times G$$

$$1 \times m \downarrow \qquad \qquad \downarrow m \times 1$$

$$G \times G \xrightarrow{m} G \times G$$



▶ Other internal algebras can be suitably defined by replacing the underlying set by an object, the operations by morphisms, and axioms by commutative diagrams.

## Split extension cores for internal semi-abelian algebras

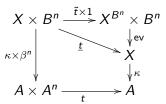
▶ Given a split extension (1) and a monomorphism  $m: S \to X$  we construct the object  $\bar{X}$  (essentially copying what happens for sets) as the meet (wide pullback) of the family of monomorphisms  $u_t: X_t \to X$ , indexed by unary ideal terms t, obtained via pullback

$$X_{t} \xrightarrow{u_{t}} X$$

$$\downarrow_{\tilde{t}} X$$

$$S^{B^{n}} \xrightarrow{m^{B^{n}}} X^{B^{n}}$$

where  $ilde{t}$  is the unique morphism making the diagram



## Split extension cores for internal semi-abelian algebras

#### **Theorem**

Let  $\mathcal V$  be a variety of semi-abelian algebras and let  $\mathbb C$  be a small complete cartesian closed category with small subobject lattice. If  $\mathcal V$  admits split extension cores, then the category of internal  $\mathcal V$  algebras in  $\mathbb C$  admits split extension cores.

### Idea of the proof

- Show that construction given for  $\bar{X}$  works in the case where  $\mathbb{C} = \mathbf{Set}^{\mathbb{X}}$  for a category  $\mathbb{X}$ .
- Use the fact that the yoneda embeding  $Y:\mathbb{C}\to \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$  preserves limits exponents and is fully faithful to show that  $\overline{Y(X)}=Y(\bar{X})$  and use this to prove that  $\bar{X}$  is a sub-algebra of X.

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