Lie group analysis of the new extended KP equation

Mr. Yanga Gaxela Prof. Masood Khalique

Faculty of Natural And Agriculture Science
Department of Mathematical and Statistical Sciences
North-West University, Mafikeng Campus

07 December 2022

Preamble

In this talk we study the new extended Kadomtsev-Petviashvili (eKP) equation [1]

$$6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6{u_x}^2 + u_{xxxx} - u_{yy} = 0. \qquad (1)$$

(1) We first compute its Lie point symmetries.

(2) We then present group-invariant solutions of the equation.

(3) Moreover, we derive conservation laws of the equation.

(4) Finally, Concluding remarks are presented.

Introduction

The standard KP equation reads

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{2}$$

It is one of the most extensively researched integrable equations in (2+1)-dimensions.

It was proposed to deal with slowly varying perturbation wave in dispersion media.

KP equation has soliton solutions through the inverse scattering transform.

Lie point symmetries

Lie point symmetries

eKP equation (1) admits the one-parameter Lie group of transformations with infinitesimal generator (Olver 1993)

$$\mathbf{X} = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \psi(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u},$$

if and only if

$$\mathbf{X}^{[4]}\Delta|_{\Delta=0} = 0, \tag{3}$$

where

$$\Delta \equiv 6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6u_x^2 + u_{xxxx} - u_{yy},$$

and

$$\mathbf{X}^{[4]} = \mathbf{X} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} + \zeta_{xxxx} \frac{\partial}{\partial u_{xxxx}}.$$

Here ζ_x, ζ_{tx}, ζ_{ty}, ζ_{tt}, ζ_{xx}, ζ_{yy} and ζ_{xxxx} are determined by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\psi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi),\\
\zeta_{ty} &= D_y(\zeta_t) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi),\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi) - u_{ty} D_t(\psi),\\
\zeta_{yy} &= D_y(\zeta_y) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi),\\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{txxx} D_x(\tau) - u_{xxxx} D_x(\xi) - u_{xxxy} D_x(\psi),
\end{aligned} \tag{4}$$

where D_t, D_x and D_y are the total derivatives described as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots , \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots , \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \cdots . \end{aligned} \tag{5}$$

Expanding equation (3) and splitting it on derivatives of u gives

$$\begin{aligned}
&\tau_x = 0,\ \tau_u = 0,\ \xi_u = 0,\ \psi_x = 0,\ \psi_u = 0,\ \eta_{uu} = 0,\ 2\eta_{xu} - 3\xi_{xx} = 0,\\
&2\tau_y + 2\alpha\xi_x - \alpha\tau_t = 0, 2\xi_y - \alpha\xi_t - (\beta+1)\psi_t = 0,\\
&\alpha\eta_u - 4\tau_y + 2\alpha(\tau_t - \xi_x) = 0,\ \alpha(-2\psi_y + 2\tau_t + \alpha\psi_t) - 4\tau_y = 0,\\
&8\eta_{yu} - 4\psi_{yy} - \alpha(4\eta_{tu} - 4\psi_{ty} + \alpha\psi_{tt}) = 0,\\
&\alpha\left(\xi_t\alpha^2 - 2\xi_y\alpha - 2(\beta+1)\xi_x + 2\beta\tau_t + 2\tau_t\right) - 8(\beta+1)\tau_y = 0,\\
&6\alpha\eta - 24u\tau_y - \alpha(12u\xi_x - 5\xi_{xxx} - 12u\tau_t + \beta\xi_t + \xi_t) = 0,\\
&4\eta_{ty}\alpha - \eta_{tt}\alpha^2 - 4\eta_{yy} + 24u\eta_{xx} + 4\eta_{xxxx} + 4\beta\eta_{tx} + 4\eta_{tx} = 0,\\
&\tau_{tt}\alpha^2 - 2\eta_{tu}\alpha^2 + 4\eta_{yu}\alpha - 4\tau_{ty}\alpha + 4\tau_{yy} + 6\beta\xi_{xx} + 6\xi_{xx} = 0,\\
&\xi_{tt}\alpha^2 - 4\xi_{ty}\alpha + 4\xi_{yy} + 48\eta_x + 48u\xi_{xx} + 20\xi_{xxxx} + 4\beta\eta_{tu}\\
&+ 4\eta_{tu} - 4\beta\xi_{tx} - 4\xi_{tx} = 0.
\end{aligned} \tag{6}$$

Thus solving the above system of PDEs yields

$$
\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{X}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{X}_3 &= \frac{\partial}{\partial y}, \\
\mathbf{X}_4 &= 6\alpha^2 t \frac{\partial}{\partial t} + \left(3\alpha^2 x - 6\beta t - 6t\right) \frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right) \frac{\partial}{\partial y} \\
&\quad - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right) \frac{\partial}{\partial u}, \\
\mathbf{X}_5 &= \left(24\alpha t^2 + 12\alpha^2 t y\right) \frac{\partial}{\partial t} + \left(12\alpha t x - 12\beta t y - 12 t y + 6\alpha^2 x y\right) \frac{\partial}{\partial x} \\
&\quad + \left(12\alpha t y - 12 t^2 + 9\alpha^2 y^2\right) \frac{\partial}{\partial y} + \left(2\alpha\beta x - 24\alpha t u - 12\alpha^2 u y\right. \\
&\quad \left.+ 2\alpha x - 2\beta^2 y - 4\beta y - 2y\right) \frac{\partial}{\partial u}.
\end{aligned}
$$

Group-Invariant Solutions and Symmetry reductions

Construction of Group-invariant solutions

Case 1. **Case 1.** We consider a linear combination

$$\mathbf{X} = \mathbf{X_1} + a\mathbf{X_2} + b\mathbf{X_3}, \tag{7}$$

where a and b are coefficients of $\mathbf{X_2}$ and $\mathbf{X_3}$.

X yields the following three invariants

$$p = x - at, \;\; q = y - bt, \;\; u = \Phi(p, q). \tag{8}$$

These invariants reduced equation (1) into a NLPDE of the form

$$\Phi_{pppp} - \left(a + \frac{1}{4}\alpha^2 a^2 + a\beta\right)\Phi_{pp} + 6\Phi\Phi_{pp} - \left(b + a\alpha + b\beta + \frac{1}{2}ab\alpha^2\right)\Phi_{pq}$$

$$- \left(1 + \frac{1}{4}b^2\alpha^2 + b\alpha\right)\Phi_{qq} = 0, \tag{9}$$

where Φ is a function of p and q. Equation (9) have the following symmetries:

$$
\begin{aligned}
\Gamma_1 =& \frac{\partial}{\partial p}, \quad \Gamma_2 = \frac{\partial}{\partial q}, \\
\Gamma_3 =& \left\{3(b\alpha + 2)^2 p + (3ab\alpha^2 + 6a\alpha + 6b\beta + 6b)q\right\} \frac{\partial}{\partial p} \\
& + 6(b\alpha + 2)^2 q \frac{\partial}{\partial q} + \Big(2ab\alpha\beta - 6b^2\alpha^2 u - b^2\beta^2 - 2b^2\beta - b^2 \\
& - 24b\alpha u + 2ab\alpha + 4a\beta - 24u\Big) \frac{\partial}{\partial \Phi}.
\end{aligned}
$$

Linear combination of translation symmetries Γ_1 and Γ_2, written as

$$\Gamma = \Gamma_1 + c\Gamma_2. \tag{10}$$

Solving the Lie characteristic equations associated with (10) we obtain two invariants

$$z = q - cp, \; \Phi = \Psi(z), \tag{11}$$

which reduce equation (9) to the fourth order NLODE

$$c^4\Psi''''(z) + A\Psi''(z) + 6c^2\left\{\Psi'^2(z) + \Psi(z)\Psi''(z)\right\} = 0 \tag{12}$$

with $z = (ac - b)t - cx + y$.

Integrating the resultant NLODE, we have

$$\frac{1}{2}c^4\Psi'^2 + \frac{1}{2}A\Psi^2 + c^2\Psi^3 + \mathbf{k_1}\Psi + \mathbf{k_2} = 0, \tag{13}$$

where $\mathbf{k_2}$ is an integration constant.

Solution of 1 by direct integration

We find the solution of (1) and express it in terms of the Jacobi elliptic function. To gain this solution, we focus on the NLODE (13). Firstly, we rewrite (13) in the form

$$\Psi'^2 + \frac{2}{c^2}\Psi^3 + \frac{A}{c^4}\Psi^2 + \frac{2\mathbf{k_1}}{c^4}\Psi + \frac{2\mathbf{k_2}}{c^4} = 0. \tag{14}$$

Suppose that v_1, v_2 and v_3 are real roots $(v_1 > v_2 > v_3)$ of the cubic equation

$$\Psi^3 + \frac{A}{2c^2}\Psi^2 + \frac{\mathbf{k_1}}{c^2}\Psi + \frac{\mathbf{k_2}}{c^2} = 0 \tag{15}$$

that satisfy the conditions

$$v_1 v_2 v_3 = -\frac{\mathbf{k_2}}{c^2}, \;\; v_1 v_2 + v_1 v_3 + v_2 v_3 = \frac{\mathbf{k_1}}{c^2}, \;\; v_1 + v_2 + v_3 = -\frac{A}{2c^2},$$

then equation (14) is written as

$$\Psi'^2 = -\frac{2}{c^2}(\Psi - v_1)(\Psi - v_2)(\Psi - v_3). \tag{16}$$

Hence, (16) has the solution

$$\Psi(z) = v_2 + (v_1 - v_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{v_1 - v_3}{2c^2}}\,(z - z_0)\,\middle|\,K^2\right\}, K^2 = \frac{v_1 - v_2}{v_1 - v_3}, \tag{17}$$

where z_0 is a constant and cn is the Jacobi cosine function. Thus by reverting to the original variables, we obtain the solution of (1) as

$$u(t, x, y) = v_2 + (v_1 - v_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{v_1 - v_3}{2c^2}}\,(z - z_0)\,\middle|\,K^2\right\}, K^2 = \frac{v_1 - v_2}{v_1 - v_3}, \tag{18}$$

Figure 1. demonstrates the solution (18) for the values $a = -4, b = -0.2, c = 1.6, t = -14, v_1 = 60, v_2 = 20.05, v_3 = -60, z_0 = 0.$

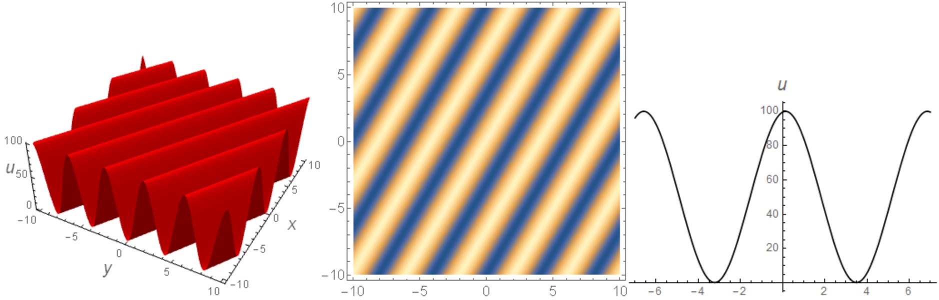

Figure: The 3D and 2D solution profiles of (18).

Kudryashov's method

We now solve the fourth order NLODE (12) using the Kudryashov's technique. To utilize the method, we begin by assuming the solution of (12) to be of the form

$$\Psi(z) = \sum_{i=0}^{M} a_i Y^i(z), \tag{19}$$

where $Y(z)$ satisfies the Riccati equation

$$Y'(z) = Y^2(z) - Y(z) \tag{20}$$

whose solution is

$$Y(z) = \frac{1}{1+e^z}. \tag{21}$$

Applying the balancing procedure to (12), we get $M = 2$. Thus, the solution (19) can be written as

$$\Psi(z) = a_0 + a_1 Y(z) + a_2 Y^2(z). \tag{22}$$

Substituting the value of $\Psi(z)$ into equation (12) and making use of (20), we get

$$
\begin{aligned}
& c^4 a_1 Y(z) + A a_1 Y(z) + 72\, c^2 a_1 Y^5(z) a_2 - 126\, c^2 a_1 Y^4(z) a_2 \\
& + 54\, c^2 a_1 Y^3(z) a_2 + 12\, c^2 a_1 Y^3(z) a_0 - 18\, c^2 a_1 Y^2(z) a_0 \\
& + 6\, c^2 a_1 Y(z) a_0 + 36\, c^2 a_2 Y^4(z) a_0 - 60\, c^2 a_2 Y^3(z) a_0 - 3\, A a_1 Y^2(z) \\
& + 24\, c^2 a_2 Y^2(z) a_0 - 60\, c^4 a_1 Y^4(z) + 24\, c^4 a_1 Y^5(z) - 15\, c^4 a_1 Y^2(z) \\
& + 50\, c^4 a_1 Y^3(z) + 120\, c^4 a_2 Y^6(z) - 336\, c^4 a_2 Y^5(z) - 30\, c^2 {a_1}^2 Y^3(z) \qquad (23) \\
& + 330\, c^4 a_2 Y^4(z) - 130\, c^4 a_2 Y^3(z) + 16\, c^4 a_2 Y^2(z) + 2\, A a_1 Y^3(z) \\
& + 6\, A a_2 Y^4(z) - 10\, A a_2 Y^3(z) + 4\, A a_2 Y^2(z) + 18\, c^2 {a_1}^2 Y^4(z) \\
& + 12\, c^2 {a_1}^2 Y^2(z) + 60\, c^2 {a_2}^2 Y^6(z) - 108\, c^2 {a_2}^2 Y^5(z) \\
& + 48\, c^2 {a_2}^2 Y^4(z) = 0.
\end{aligned}
$$

After splitting (23) with respect to the like powers of $Y(z)$, we have

$$Y^6(z): \; 2\,c^4 a_2 + \; c^2 a_2{}^2 = 0,$$
$$Y^5(z) \; : 2c^4 a_1 - 24\,c^4 a_2 + 6\,c^2 a_1 a_2 - 9\,c^2 a_2{}^2 = 0, \quad (24)$$
$$Y^4(z): \; 55\,c^4 a_2 - 10\,c^4 a_1 + 6\,c^2 a_0 a_2 + 3\,c^2 a_1{}^2 - 21\,c^2 a_1 a_2 + 8\,c^2 a_2{}^2 +$$
$$Y^3(z): \; 25\,c^4 a_1 - 65\,c^4 a_2 + 6\,c^2 a_0 a_1 - 30\,c^2 a_0 a_2 - 15\,c^2 a_1{}^2 + 27\,c^2 a_1 a$$
$$+ \; A a_1 - 5\,A a_2 = 0,$$
$$Y^2(z): \; 16\,c^4 a_2 - 15\,c^4 a_1 - 18\,c^2 a_0 a_1 + 24\,c^2 a_0 a_2 + 12\,c^2 a_1{}^2 - 3\,A a_1 +$$
$$Y(z): \; c^4 a_1 + 6\,c^2 a_0 a_1 + A a_1 = 0.$$

$$a_0 = -\frac{c^4 + A}{c^2}, \; a_1 = 2\,c^2, \; a_2 = -2\,c^2. \tag{25}$$

Thus the solution (22) is written as

$$\Psi(z) = -\frac{c^4 + A}{c^2} + \frac{2\,c^2}{1 + e^z} - 2\,c^2 \left(\frac{1}{1 + e^z}\right)^2.$$

Therefore, reverting to the original variables, we get

$$u(t, x, y) = -\frac{c^4 + A}{c^2} + \frac{2\,c^2}{1 + e^z} - 2\,c^2 \left(\frac{1}{1 + e^z}\right)^2, \tag{26}$$

where $z = (ac - b)t - cx + y$

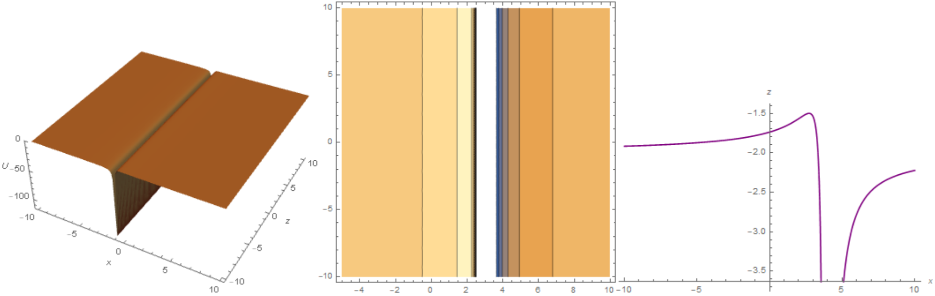

Figure: The 3D and 2D solution profiles of (26).

Figure 2. demonstrates the solution (26) for the values $a = 0.9, b = 1.5, c = -1, A = 1, t = 1, y = 3.$

Conservation laws

Ibragimov's theorem

We now construct conservation laws for the eKP equation (1) by employing the Ibragimov's theorem. Firstly we write the adjoint

equation in the form

$$F^* \equiv \frac{\delta}{\delta u}\left\{v\left(6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6u_x^2 + u_{xxxx} - u_{yy}\right)\right\} = \tag{27}$$

Expanding (27), we obtain the adjoint equation as

$$6v_{xx}u - \frac{1}{4}\alpha^2 v_{tt} + (\beta + 1)v_{tx} + \alpha v_{ty} + v_{xxxx} - v_{yy} = 0. \tag{28}$$

Equation (1) and its adjoint (28) have a second-order Lagrangian

$$\mathcal{L} = v\big(6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6{u_x}^2 - u_{yy}\big) + v_{xx}u_{xx}. \tag{29}$$

Applying Ibragimov's theorem [2], we deduce that conserved vectors

$$
\begin{aligned}
C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\
+ D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots \right] + D_j D_k(W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots, \quad (30)
\end{aligned}
$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, \ldots, m$ is the Lie characteristic function and $\mathcal{L}$ is the Lagrangian.

We have five cases.
Case1. Let us consider Lie point symmetry $\mathbf{X_1}$.
The Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_1}$ is

$$\begin{aligned}
C_1^t =& \frac{1}{2}\alpha u_{ty} v + \frac{1}{2}\beta u_{tx} v + 6{u_x}^2 v - u_{yy} v + 6u_{xx} u v + \frac{1}{2} u_{tx} v - \frac{1}{4}\alpha^2 u_t v_t \\
&+ \frac{1}{2}\beta u_t v_x + \frac{1}{2} u_t v_x + \frac{1}{2}\alpha u_t v_y + u_{xx} v_{xx}, \\
C_1^x =& 6u_t v_x u - \frac{1}{2}\beta u_{tt} v - 6u_t u_x v - 6u_{tx} u v - \frac{1}{2} u_{tt} v + \frac{1}{2}\beta u_t v_t \\
&+ u_t v_{xxx} + v_t u_{xxx} - v_{xx} u_{tx} - u_{xx} v_{tx} + \frac{1}{2} u_t v_t, \\
C_1^y =& \frac{1}{2}\alpha u_t v_t - \frac{1}{2}\alpha u_{tt} v + u_{ty} v - u_t v_y.
\end{aligned}$$

Case 2. For symmetry $\mathbf{X_2} = \partial/\partial x$, the Lie characteristic functions are $W^1 = -u_x$ and $W^2 = -v_x$. Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_2}$ is

$$
\begin{aligned}
C_2^t =& \frac{1}{4}\alpha^2 u_{tx} v - \frac{1}{2}\alpha u_{xy} v - \frac{1}{2}\beta u_{xx} v - \frac{1}{2} u_{xx} v - \frac{1}{4}\alpha^2 v_t u_x + \frac{1}{2}\beta u_x v_x \\
&+ \frac{1}{2}\alpha u_x v_y + \frac{1}{2} u_x v_x, \\
C_2^x =& \alpha u_{ty} v + \frac{1}{2}\beta u_{tx} v - u_{yy} v + 6 u_x v_x u + \frac{1}{2} u_{tx} v + \frac{1}{2} v_t u_x - \frac{1}{4}\alpha^2 u_{tt} v \\
&+ \frac{1}{2}\beta v_t u_x - u_{xx} v_{xx} + u_{xxx} v_x + u_x v_{xxx}, \\
C_2^y =& u_{xy} v - \frac{1}{2}\alpha u_{tx} v + \frac{1}{2}\alpha v_t u_x - u_x v_y .
\end{aligned}
$$

Case 3. For symmetry $\mathbf{X_3} = \partial/\partial y$, the Lie characteristic functions are $W^1 = -u_y$ and $W^2 = -v_y$. Hence, the conserved vector corresponding to the Lie point symmetry $\mathbf{X_3}$ is

$$
\begin{aligned}
C_3^t =& \frac{1}{4}\alpha^2 u_{ty} v - \frac{1}{2}\alpha u_{yy} v - \frac{1}{2}\beta u_{xy} v - \frac{1}{2} u_{xy} v - \frac{1}{4}\alpha^2 v_t u_y + \frac{1}{2}\beta u_y v_x \\
&+ \frac{1}{2} u_y v_x + \frac{1}{2}\alpha u_y v_y, \\
C_3^x =& 6 u_y v_x u - \frac{1}{2}\beta u_{ty} v - 6 u_x u_y v - 6 u_{xy} u v - \frac{1}{2} u_{ty} v + \frac{1}{2}\beta v_t u_y + \frac{1}{2} v_t u_y \\
&- u_{xx} v_{xy} + u_{xxx} v_y - v_{xx} u_{xy} + u_y v_{xxx}, \\
C_3^y =& \frac{1}{2}\alpha u_{ty} v - \frac{1}{4}\alpha^2 u_{tt} v + \beta u_{tx} v + 6 {u_x}^2 v + 6 u_{xx} u v + u_{tx} v + \frac{1}{2}\alpha v_t u_y \\
&+ u_{xx} v_{xx} - u_y v_y.
\end{aligned}
$$

Case 4. For symmetry

$$\mathbf{X_4} = 6\alpha^2 t\frac{\partial}{\partial t} + \left(3\alpha^2 x - 6\beta t - 6t\right)\frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right)\frac{\partial}{\partial y} - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right)\frac{\partial}{\partial u},$$

the Lie characteristic function is

$$W^1 = -\left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right) - 6\alpha^2 t u_t - \left(3\alpha^2 x - 6\beta t - 6t\right) u_x - \left(6\alpha t + 9\alpha^2 y\right) u_y,$$

and

$$W^2 = -\left(\beta^2 + 2\beta + 6\alpha^2 v + 1\right) - 6\alpha^2 t v_t - \left(3\alpha^2 x - 6\beta t - 6t\right) v_x - \left(6\alpha t + 9\alpha^2 y\right) v_y.$$

Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_4}$ is

$$
\begin{aligned}
C_4^t = & 3\alpha^4 v u_t - \frac{3}{2}\alpha^4 u v_t - \frac{9}{4}\alpha^4 y u_y v_t - \frac{3}{4}\alpha^4 x u_x v_t - \frac{3}{2}\alpha^4 t u_t v_t + \frac{9}{4}\alpha^4 y v u_{ty} \\
& + \frac{3}{4}\alpha^4 x v u_{tx} - 6\alpha^3 v u_y + 3\alpha^3 u v_y + \frac{9}{2}\alpha^3 y u_y v_y - \frac{9}{2}\alpha^3 y v u_{yy} \\
& - \frac{3}{2}\alpha^3 x v u_{xy} + 3\alpha^3 t v_y u_t - \frac{3}{2}\alpha^3 t u_y v_t + \frac{9}{2}\alpha^3 t v u_{ty} + 36\alpha^2 t v {u_x}^2 \\
& - 9\alpha^2 t v u_{yy} - 6\beta\alpha^2 v u_x - 6\alpha^2 v u_x + 3\beta\alpha^2 u v_x + 3\alpha^2 u v_x + \frac{9}{2}\alpha^2 y u_y v_x \\
& + \frac{9}{2}\alpha^2 \beta y u_y v_x + \frac{3}{2}\alpha^2 x u_x v_x + \frac{3}{2}\alpha^2 \beta x u_x v_x - \frac{9}{2}\alpha^2 y v u_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}\beta\alpha^2 xvu_{xx}+36\alpha^2 tuvu_{xx}+6\alpha^2 tu_{xx}v_{xx}+3\alpha^2 tv_x u_t+3\beta\alpha^2 tv_x u_t\\
&-\frac{1}{4}\beta^2\alpha^2 v_t-\frac{1}{2}\beta\alpha^2 v_t+\frac{3}{2}\alpha^2 tu_x v_t+\frac{3}{2}\beta\alpha^2 tu_x v_t-\frac{1}{4}\alpha^2 v_t+\frac{3}{2}\alpha^2 tvu_{tx}\\
&+\frac{3}{2}t\beta\alpha^2 vu_{tx}+\frac{1}{2}\beta^2\alpha v_y+\beta\alpha v_y+\frac{1}{2}v_y\alpha-3\alpha tv_y u_x-3t\beta\alpha v_y u_x\\
&+3\alpha tu_y v_x+3\beta\alpha tu_y v_x+\frac{1}{2}\beta^3 v_x+\frac{3}{2}\beta^2 v_x+\frac{3}{2}\beta v_x-3\beta^2 tu_x v_x-3tu_x v_x\\
&-6\beta tu_x v_x+\frac{1}{2}v_x+3\beta^2 tvu_{xx}+3tvu_{xx}+6\beta tvu_{xx}-\frac{3}{2}\alpha^2 xvu_{xx}\\
&+\frac{3}{2}\alpha^3 xv_y u_x+3\alpha^2 tu_y v_y,
\end{aligned}
$$

$$
\begin{aligned}
C_4^x = {} & 3\alpha^3 xvu_{ty} - \frac{3}{4}\alpha^4 xvu_{tt} - 3\alpha^2 xvu_{yy} - 90\alpha^2 uvu_x - 54\alpha^2 yvu_y u_x \\
& + 36\alpha^2 u^2 v_x + 18\alpha^2 xuu_x v_x - 54\alpha^2 yuvu_{xy} - 3\alpha^2 v_x u_{xx} \\
& - 9\alpha^2 u_x v_{xx} - 9\alpha^2 yu_{xy} v_{xx} - 3\alpha^2 xu_{xx} v_{xx} + 9\alpha^2 yv_y u_{xxx} \\
& + 6\alpha^2 uv_{xxx} + 9\alpha^2 yu_y v_{xxx} + 3\alpha^2 xu_x v_{xxx} - 6\beta\alpha^2 vu_t - 6\alpha^2 vu_t \\
& - 36\alpha^2 tvu_x u_t + 36\alpha^2 tuv_x u_t + 6\alpha^2 tv_{xxx} u_t + 3\beta\alpha^2 uv_t + 3\alpha^2 uv_t \\
& + \frac{9}{2}\beta\alpha^2 yu_y v_t + \frac{3}{2}\alpha^2 xu_x v_t + \frac{3}{2}\beta\alpha^2 xu_x v_t + 6tu_{xxx} v_t\alpha^2 + 3tu_t v_t\alpha^2 \\
& - \frac{9}{2}\alpha^2 yvu_{ty} - \frac{9}{2}\beta\alpha^2 yvu_{ty} + \frac{3}{2}\alpha^2 xvu_{tx} + \frac{3}{2}\beta\alpha^2 xvu_{tx} - 36\alpha^2 tuvu_{tx} \\
& - \frac{3}{2}\alpha^2 tvu_{tt} - \frac{3}{2}\beta\alpha^2 tvu_{tt} - 3\beta\alpha vu_y - 3\alpha vu_y - 36\alpha tvu_y u_x
\end{aligned}
$$

$$
\begin{aligned}
&- 6\alpha t v_{xy} u_{xx} - 6\alpha t u_{xy} v_{xx} + 6\alpha t v_y u_{xxx} + 6\alpha t u_y v_{xxx} + 3\alpha t u_y v_t \\
&+ 6 t v u_{yy} + 6\beta t v u_{yy} - 3\beta^2 v u_x - 6\beta v u_x - 3 v u_x + 6\beta^2 u v_x + 12\beta u v_x \\
&- 36 t u u_x v_x - 36\beta t u u_x v_x + 6 t u_{xx} v_{xx} + 6\beta t u_{xx} v_{xx} - 6 t v_x u_{xxx} - 6\beta t v_x u_{xxx} \\
&+ 2\beta v_{xxx} - 6 t u_x v_{xxx} - 6\beta t u_x v_{xxx} + v_{xxx} + \frac{1}{2}\beta^3 v_t + \frac{3}{2}\beta^2 v_t + \frac{3}{2}\beta v_t \\
&- 3 t u_x v_t - 6\beta t u_x v_t + \frac{1}{2} v_t - 3\beta^2 t v u_{tx} - 3 t v u_{tx} - 6\beta t v u_{tx} - 9\beta\alpha t v u_{ty} \\
&- 6\alpha^2 t u_{xx} v_{tx} + 3\beta\alpha^2 t u_t v_t + 54\alpha^2 y u u_y v_x + \beta^2 v_{xxx} - 9\alpha t v u_{ty} \\
&- 36\alpha t u v u_{xy} - 6\alpha^2 t v_{xx} u_{tx} - 3\beta^2 t u_x v_t + 6 u v_x - 9\alpha^2 y v_{xy} u_{xx} \\
&+ 3\alpha^2 x v_x u_{xxx} + \frac{9}{2}\alpha^2 y u_y v_t + 36\alpha t u u_y v_x + 3\beta\alpha t u_y v_t,
\end{aligned}
$$

$$
\begin{aligned}
C_4^y = & 3\alpha^3 u v_t - \frac{9}{4}\alpha^4 y v u_{tt} - 6\alpha^3 v u_t + \frac{9}{2}\alpha^3 y u_y v_t + \frac{3}{2}\alpha^3 x u_x v_t + 3\alpha^3 t u_t v_t \\
& - \frac{3}{2}\alpha^3 x v u_{tx} - \frac{9}{2}\alpha^3 t v u_{tt} + 54\alpha^2 y v u_x^2 + 12\alpha^2 v u_y - 6\alpha^2 u v_y \\
& + 3\alpha^2 x v u_{xy} + 54\alpha^2 y u v u_{xx} + 9\alpha^2 y u_{xx} v_{xx} - 6\alpha^2 t v_y u_t + 3\alpha^2 t u_y v_t \\
& + 9\alpha^2 t v u_{ty} + 9\alpha^2 y v u_{tx} + 9\beta\alpha^2 y v u_{tx} + 36\alpha t v u_x^2 - 6\alpha t u_y v_y \\
& + 36\alpha t u v u_{xx} + 6\alpha t u_{xx} v_{xx} + \frac{1}{2}\beta^2 \alpha v_t + \beta\alpha v_t - 3\alpha t u_x v_t - 3t\beta\alpha u_x v_t \\
& + 9\alpha t v u_{tx} + 9\beta\alpha t v u_{tx} - \beta^2 v_y - 2\beta v_y - v_y + 6 t v_y u_x + 6\beta t v_y u_x \\
& - 3\alpha^2 x v_y u_x + \frac{9}{2}\alpha^3 y v u_{ty} + 3\alpha v u_x - 9\alpha^2 y u_y v_y - 6t\beta v u_{xy} \\
& + \frac{1}{2}\alpha v_t + 3\beta\alpha v u_x - 6 t v u_{xy}.
\end{aligned}
$$

Case 5. Finally for the symmetry

$$\begin{aligned} \mathbf{X_5} = & \left(24\alpha t^2 + 12\alpha^2 ty\right)\frac{\partial}{\partial t} + \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right)\frac{\partial}{\partial x} \\ & + \left(12\alpha ty - 12t^2 + 9\alpha^2 y^2\right)\frac{\partial}{\partial y} + \left(2\alpha\beta x - 24\alpha tu - 12\alpha^2 uy\right. \\ & \left. + 2\alpha x - 2\beta^2 y - 4\beta y - 2y\right)\frac{\partial}{\partial u}, \end{aligned}$$

the Lie characteristic function is

$$\begin{aligned} W^1 = & \left(2\alpha\beta x - 24\alpha tu - 12\alpha^2 uy + 2\alpha x - 2\beta^2 y - 4\beta y - 2y\right) - \left(24\alpha t^2\right. \\ & \left. + 12\alpha^2 ty\right) u_t - \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right) u_x - (12\alpha ty \\ & -12t^2 + 9\alpha^2 y^2\Big) u_y, \end{aligned}$$

and

$$W^2 = \left(2\alpha\beta x - 24\alpha t v - 12\alpha^2 v y + 2\alpha x - 2\beta^2 y - 4\beta y - 2y\right) - \left(24\alpha t^2 + 12\alpha^2 t y\right) v_t - \left(12\alpha t x - 12\beta t y - 12 t y + 6\alpha^2 x y\right) v_x - (12\alpha t y - 12 t^2 + 9\alpha^2 y^2\Big) v_y.$$

Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_5}$ is

$$C_5^t = 6\alpha^4 y v u_t - 3\alpha^4 y u v_t - \frac{9}{4}\alpha^4 y^2 u_y v_t - \frac{3}{2}\alpha^4 x y u_x v_t - 3\alpha^4 t y u_t v_t - 12\alpha^3 y v u_y + 6\alpha^3 y u v_y + \frac{9}{2}\alpha^3 y^2 u_y v_y - \frac{9}{2}\alpha^3 y^2 v u_{yy} + 3\alpha^3 x y v_y u_x + 6\alpha^3 t y v_y u_t + \frac{1}{2}\alpha^3 x v_t + \frac{1}{2}\alpha^3 \beta x v_t - 6\alpha^3 t u v_t - 3\alpha^3 t y u_y v_t - 3\alpha^3 t x u_x v_t + 9\alpha^3 t y v u_{ty} + 3\alpha^3 t x v u_{tx} + 72\alpha^2 t y v u_x^2 - 24\alpha^2 t v u_y$$

$$
\begin{aligned}
&+6\alpha^2 tyu_y v_y - 18\alpha^2 tyvu_{yy} - 12\alpha^2 yvu_x - 12\alpha^2\beta yvu_x + 6\alpha^2 txv_y u_x \\
&+6\alpha^2\beta yuv_x + \frac{9}{2}\alpha^2 y^2 u_y v_x + \frac{9}{2}\alpha^2\beta y^2 u_y v_x + 3\alpha^2 xyu_x v_x + 3\alpha^2\beta xyu_x v_x \\
&-6\alpha^2 txvu_{xy} - \frac{9}{2}\alpha^2\beta y^2 vu_{xy} - 3\alpha^2 xyvu_{xx} - 3\alpha^2\beta xyvu_{xx} + 72\alpha^2 tyuvu_{xx} \\
&+12\alpha^2 tyu_{xx} v_{xx} + 12\alpha^2 t^2 v_y u_t + 6\alpha^2 tyv_x u_t + 6\alpha^2\beta tyv_x u_t - \frac{1}{2}\alpha^2\beta^2 yv_t \\
&-\alpha^2\beta yv_t + 3\alpha^2 t^2 u_y v_t + 3\alpha^2 tyu_x v_t + 3\alpha^2\beta tyu_x v_t + 9\alpha^2 t^2 vu_{ty} + 3\alpha^2 tyvu_{tx} \\
&+144\alpha t^2 vu_x^2 + \alpha\beta^2 yv_y + \alpha yv_y + 2\alpha\beta yv_y - 6\alpha t^2 u_y v_y - 18\alpha t^2 vu_{yy} \\
&-6\alpha tyv_y u_x - 6\alpha\beta tyv_y u_x - \beta^2\alpha xv_x - \alpha xv_x - 2\alpha\beta xv_x + 12\alpha tuv_x \\
&+6\alpha\beta tyu_y v_x + 6\alpha txu_x v_x + 6\alpha\beta txu_x v_x - 6\alpha txvu_{xx} - 6\alpha\beta txvu_{xx} \\
&+24\alpha t^2 u_{xx} v_{xx} + 12\alpha t^2 v_x u_t + 12\alpha\beta t^2 v_x u_t + 12\alpha t^2 vu_{tx} + 12\alpha\beta t^2 vu_{tx}
\end{aligned}
$$

$$
\begin{aligned}
&+ yv_x + 3\beta yv_x - 6t^2 u_y v_x - 6\beta t^2 u_y v_x - 6\beta^2 tyu_x v_x - 6tyu_x v_x \\
&+ 6t^2 vu_{xy} + 6\beta t^2 vu_{xy} + 6\beta^2 tyvu_{xx} + 6tyvu_{xx} + 12\beta tyvu_{xx} - 12\alpha\beta tvu_x \\
&+ \frac{3}{2}\alpha^4 xyvu_{tx} + 12\alpha^3 tvu_t - 6\alpha^3 t^2 u_t v_t + 12\alpha^2 tuv_y - \frac{9}{2}\alpha^2 y^2 vu_{xy} \\
&+ 3\alpha^2\beta tyvu_{tx} + 3\beta^2 yv_x + 144\alpha t^2 uvu_{xx} - \frac{1}{2}\alpha^2 yv_t - 12\alpha tvu_x \\
&+ 6\alpha^2 yuv_x + \beta^3 yv_x - 3\alpha^3 xyvu_{xy} + 6\alpha tyu_y v_x + \frac{9}{4}\alpha^4 y^2 vu_{ty} + 12\beta\alpha tuv_x \\
&- 12\beta tyu_x v_x - \alpha^2 xv_y - \alpha^2\beta xv_y,
\end{aligned}
$$

$$
\begin{aligned}
C_5^x = & 6\alpha^3 xyvu_{ty} - \frac{3}{2}\alpha^4 xyvu_{tt} - 3\alpha^3 txvu_{tt} - 6\alpha^2 xyvu_{yy} - 180\alpha^2 yuvu_x \\
& + 72\alpha^2 yu^2 v_x + 54\alpha^2 y^2 uu_y v_x + 36\alpha^2 xyuu_x v_x - 54\alpha^2 y^2 uvu_{xy} \\
& - 9\alpha^2 y^2 v_{xy} u_{xx} - 18\alpha^2 yu_x v_{xx} - 9\alpha^2 y^2 u_{xy} v_{xx} - 6\alpha^2 xyu_{xx} v_{xx} \\
& + 12\alpha^2 yuv_{xxx} + 9\alpha^2 y^2 u_y v_{xxx} + 6\alpha^2 xyu_x v_{xxx} - 12\alpha^2 yvu_t \\
& + 72\alpha^2 tyuv_x u_t + 12\alpha^2 tyv_{xxx} u_t + 6\alpha^2 yuv_t + 6\alpha^2 \beta yuv_t + \frac{9}{2}\alpha^2 y^2 u_y v_t \\
& + 3\alpha^2 xyu_x v_t + 3\alpha^2 \beta xyu_x v_t + 12\alpha^2 tyu_{xxx} v_t + 6\alpha^2 tyu_t v_t \\
& + 12\alpha^2 txvu_{ty} - \frac{9}{2}\alpha^2 \beta y^2 vu_{ty} + 3\alpha^2 xyvu_{tx} + 3\alpha^2 \beta xyvu_{tx} \\
& - 72\alpha^2 tyuvu_{tx} + 6\alpha^2 \beta tyu_t v_t - 12\alpha^2 \beta yvu_t - 6\alpha^2 yv_x u_{xx}
\end{aligned}
$$

$$
\begin{aligned}
&-12\alpha^2 tyu_{xx}v_{tx} - 3\alpha^2 tyvu_{tt} - 3\alpha^2\beta tyvu_{tt} - 6\alpha yvu_y - 6\alpha\beta yvu_y \\
&+ 6\alpha xvu_x + 6\alpha\beta xvu_x - 360\alpha tuvu_x - 72\alpha tyvu_y u_x + 144tu^2\alpha v_x \\
&+ 72\alpha tyuu_y v_x + 72\alpha txuu_x v_x - 72\alpha tyuvu_{xy} - 12\alpha tv_x u_{xx} - 12\alpha tyv_{xy}u_{xx} \\
&- 36\alpha tu_x v_{xx} - 12\alpha tyu_{xy}v_{xx} - 12\alpha txu_{xx}v_{xx} + 2\alpha v_{xx} + 12\alpha tyv_y u_{xxx} \\
&- 2\alpha xv_{xxx} - 2\alpha\beta xv_{xxx} + 24\alpha tuv_{xxx} + 12\alpha tyu_y v_{xxx} + 12\alpha txu_x v_{xxx} \\
&- 144\alpha t^2 vu_x u_t + 144\alpha t^2 uv_x u_t + 24\alpha t^2 v_{xxx}u_t - \alpha\beta^2 xv_t - \alpha xv_t \\
&+ 12\alpha\beta tuv_t + 6\alpha tyu_y v_t + 6\alpha\beta tyu_y v_t + 6\alpha txu_x v_t + 6\alpha\beta txu_x v_t \\
&+ 12\alpha\beta t^2 u_t v_t - 18\alpha tyvu_{ty} - 18\alpha\beta tyvu_{ty} + 6\alpha txvu_{tx} + 6\alpha\beta txvu_{tx} \\
&- 24\alpha t^2 v_{xx}u_{tx} - 24\alpha t^2 u_{xx}v_{tx} - 12\alpha t^2 vu_{tt} - 12\alpha\beta t^2 vu_{tt} + 12tvu_y \\
&+ 12\beta tyvu_{yy} - 6\beta^2 yvu_x - 6yvu_x - 12\beta yvu_x + 72t^2 vu_y u_x + 12\beta^2 yuv_x
\end{aligned}
$$

$$
\begin{aligned}
&-72t^2uu_yv_x-72tyuu_xv_x-72\beta tyuu_xv_x+72t^2uvu_{xy}+12t^2v_{xy}u_{xx}\\
&+12t^2u_{xy}v_{xx}+12tyu_{xx}v_{xx}+12\beta tyu_{xx}v_{xx}-12t^2v_yu_{xxx}-12tyv_xu_{xxx}\\
&-12\beta tyv_xu_{xxx}+2\beta^2yv_{xxx}+2yv_{xxx}+4\beta yv_{xxx}-12t^2u_yv_{xxx}\\
&+\beta^3yv_t+3\beta^2yv_t+yv_t+3\beta yv_t-6t^2u_yv_t-6\beta t^2u_yv_t-6\beta^2tyu_xv_t\\
&+6t^2vu_{ty}+6\beta t^2vu_{ty}-6\beta^2tyvu_{tx}-6tyvu_{tx}-12\beta tyvu_{tx}\\
&-54\alpha^2y^2vu_yu_x+6\alpha^2xyv_xu_{xxx}-72\alpha^2tyvu_xu_t+\frac{9}{2}\alpha^2\beta y^2u_yv_t\\
&-12\alpha^2tyv_{xx}u_{tx}-12\alpha\beta xuv_x+12\alpha txv_xu_{xxx}-12\beta tyu_xv_{xxx}+12\alpha t^2u_tv_t\\
&-12tyu_xv_{xxx}+24\beta yuv_x-36\alpha\beta tvu_t-12\beta tyu_xv_t+12tyvu_{yy}\\
&+2\beta\alpha v_{xx}+12\alpha tuv_t-144\alpha t^2uvu_{tx}-36\alpha tvu_t-12\alpha txvu_{yy}-6tyu_xv_t\\
&+12yuv_x+12\beta tvu_y-2\alpha\beta xv_t+24\alpha t^2u_{xxx}v_t+9\alpha^2y^2v_yu_{xxx}\\
&-12\alpha xuv_x-\frac{9}{2}\alpha^2y^2vu_{ty},
\end{aligned}
$$

$$
\begin{aligned}
C_5^y = {} & 6\alpha^3 yuv_t - \frac{9}{4}\alpha^4 y^2 vu_{tt} - 12\alpha^3 yvu_t + \frac{9}{2}\alpha^3 y^2 u_y v_t + 3\alpha^3 xyu_x v_t \\
& + \frac{9}{2}\alpha^3 y^2 vu_{ty} - 3\alpha^3 xyvu_{tx} - 9\alpha^3 tyvu_{tt} + 54\alpha^2 y^2 vu_x^2 + 24\alpha^2 yvu_y \\
& - 9\alpha^2 y^2 u_y v_y - 6\alpha^2 xyv_y u_x + 6\alpha^2 xyvu_{xy} + 54\alpha^2 y^2 uvu_{xx} \\
& - 12\alpha^2 tyv_y u_t - \alpha xv_t^2 - \alpha^2\beta xv_t + 12\alpha^2 tuv_t + 6\alpha^2 tyu_y v_t + 6\alpha^2 txu_x v_t \\
& + 12\alpha^2 t^2 u_t v_t + 18\alpha^2 tyvu_{ty} + 9\alpha^2 y^2 vu_{tx} - 6\alpha^2 txvu_{tx} + 9\alpha^2\beta y^2 vu_{tx} \\
& + 72\alpha tyvu_x^2 + 48\alpha tvu_y + 2\alpha xv_y + 2\alpha\beta xv_y - 24\alpha tuv_y - 12\alpha tyu_y v_y \\
& + 6\alpha\beta yvu_x - 12\alpha txv_y u_x + 12\alpha txvu_{xy} + 72\alpha tyuvu_{xx} + 12\alpha tyu_{xx} v_{xx} \\
& + \alpha\beta^2 yv_t + \alpha yv_t + 2\alpha\beta yv_t - 6\alpha t^2 u_y v_t - 6\alpha tyu_x v_t - 6\alpha\beta tyu_x v_t \\
& + 18\alpha tyvu_{tx} + 18\alpha\beta tyvu_{tx} - 72t^2 vu_x^2 + 2\beta^2 v + 4\beta v + 2v - 2\beta^2 yv_y \\
& - 4y\beta v_y + 12t^2 u_y v_y - 12tvu_x - 12\beta tvu_x + 12tyv_y u_x + 12\beta tyv_y u_x \\
& - 12\beta tyvu_{xy} - 72t^2 uvu_{xx} - 12t^2 u_{xx} v_{xx} - 12t^2 vu_{tx} - 12\beta t^2 vu_{tx}
\end{aligned}
$$

$$+6\alpha^3 tyu_t v_t - 12\alpha^2 yuv_y - 24\alpha^2 tvu_t - 9\alpha^2 t^2 vu_{tt} + 6\alpha yvu_x - 24\alpha t^2 v_y u_t$$
$$+18\alpha t^2 vu_{ty} - 2yv_y - 12tyvu_{xy} + 9\alpha^2 y^2 u_{xx} v_{xx}.$$

Case 2. We consider Lie point symmetry X_4, namely

$$X_4 = 6\alpha^2 t\frac{\partial}{\partial t} + \left(3\alpha^2 x - 6\beta t - 6t\right)\frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right)\frac{\partial}{\partial y} - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right)\frac{\partial}{\partial u}.$$

whose invariants are

$$J_1 = \frac{\alpha^2 x + 2t(\beta+1)}{\sqrt{t}\alpha^2}, J_2 = \frac{\alpha\, y + 2\, t}{t^{3/2}\alpha},$$
$$J_3 = \frac{\left(6\, u\alpha^2 + \beta^2 + 2\,\beta + 1\right) t}{6\alpha^2}.$$

thus 1 is reduced into NLPDE

$$\alpha^2 q^2 G_{qq} - 7\alpha^2 q G_q - 9\alpha^2 p^2 G_{pp} - 2\left(3\alpha^2 pq - \frac{8\beta}{\alpha} - \frac{8}{\alpha}\right) G_{pq} - 27\alpha^2 p G_p - 8\alpha^2 G(p,q) + 96 G_{qq} G(p,q) + 96 G_p^2 + 16 G_{qqqq} = 0, \quad (31)$$

Case 3. We now consider symmetry X_5,

$$\mathbf{X}_5 = \left(24\alpha t^2 + 12\alpha^2 ty\right)\frac{\partial}{\partial t} + \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right)\frac{\partial}{\partial x} + \left(12\alpha ty - 12t^2 + 9\alpha^2 y^2\right)\frac{\partial}{\partial y} + \Big(2\alpha\beta x - 24\alpha tu - 12\alpha^2 uy + 2\alpha x - 2\beta^2 y - 4\beta y - 2y\Big)\frac{\partial}{\partial u}.$$

We were unable to find invariants for this symmetry.

Concluding Remarks

In this presentation we studied the eKP equation.

We first computed the Lie symmetries of the equation.

We obtained symmetry reductions and group-invariant solutions.

Furthermore, we derived the conservation laws for the equation using the Ibragimov's theorem.

Bibliography

- Y.L. Ma, A.M. Wazwaz, B.Q. Li, New extended Kadomtsev?Petviashvili equation: multiple soliton solutions, breather, lump and interaction solutions. Nonlinear Dyn., 104 (2021) 1581?-1594.
- N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Vols 1–3, CRC Press, Boca Raton, Florida, 1994–1996.
- P. J. Morrison, Meiss JD, and Carey JR, Scattering of RLW solitary waves, physica 11D (1981), 324–336.
- P.J. Olver, Applications of Lie Groups to Differential Equations, second ed., Springer-Verlag, Berlin, 1993.
- Anco S.C., and Bluman G., "Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications." European Journal of Applied

Bibliography

- L.R.T. Gardner and G.A. Gardner, Solitary waves of the equal width wave equation, Comput.Phys, 101 (1992), 218-223.
- N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, John Wiley & Sons, Chichester, NY, 1999.

Bibliography

Y.L. Ma, A.M. Wazwaz, B.Q. Li, New extended Kadomtsev?Petviashvili equation: multiple soliton solutions, breather, lump and interaction solutions. Nonlinear Dyn., 104 (2021) 1581?-1594.

W. Sarlet. Comment on conservation laws of higher order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives, Journal of Physics A: Mathematical and Theoretical 43.45 (2010) 458001.

J. Billingham, A.C. King, Wave Motion, Cambridge University Press, Cambridge, 2000.

A. K. Khalifa, and K.R. Raslan, Finite difference methods for the equal width wave equation, Math.Sos 7 (1999), 4755.
0.15cm

Thank you so much for your attention