Non-local sectional curvature as applied to random geometric graphs, the earth and fractals

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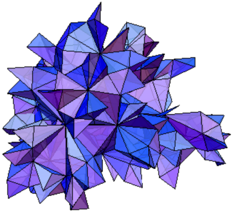
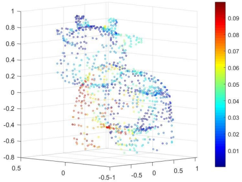
Applications of generalized curvature

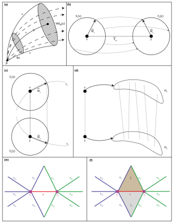
Quantum gravity, manifold reconstruction and complex network theory already use some generalized curvature estimators, inspired by and/or derived from differential geometric notions of curvature.

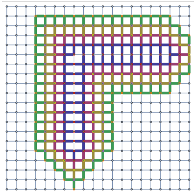


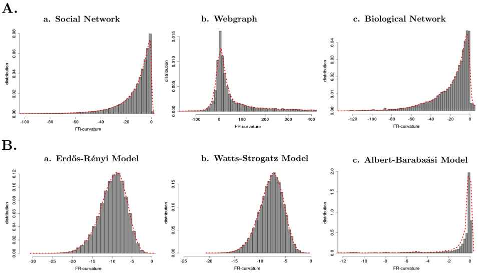


Existing generalized curvatures

Existing curvature estimators on graphs and hypergraphs focus on analogues of the Ricci curvature tensor.







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Gaps to be filled

The following gaps exist in the resurging field of network curvature

1. A standard way of investigating convergence to differential geometric curvature notions, independent of graph construction method
2. Sectional curvature
3. Manifestly mesoscopic curvature

Since sectional curvature completely determines the curvature tensor, it also captures all aspects of Ricci curvature.

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The cosine rule in spaces of constant curvature

Definition 2.1

The *sectional curvature* of a 2D linear subspace σ_p of the tangent space at a point p in a Riemannian manifold is the Gaussian curvature of the geodesic plane that is the image of σ_p under the exponential map at p.

In a 2D Riemannian manifold with constant sectional curvature K, if we have an embedded triangle with side lengths a, b, c (and the angle opposite to c is γ), we have a generalization of the cosine rule, namely

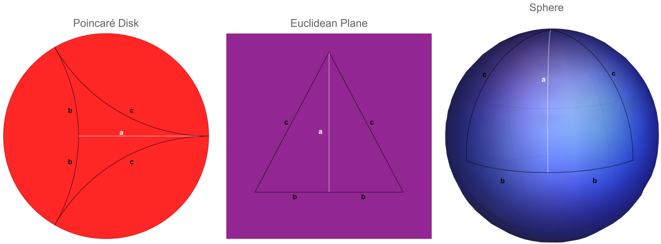
$$\cos(c\sqrt{K}) = \cos(a\sqrt{K})\cos(b\sqrt{K}) + \sin(a\sqrt{K})\sin(b\sqrt{K})\cos(\gamma)$$

Constructing right-angled triangles in constant curvature

If we can set $\gamma = \frac{\pi}{2}$, we can just consider the generalization of the Pythagorean theorem

$$\cos(c\sqrt{K}) = \cos(a\sqrt{K})\cos(b\sqrt{K}).$$

Again in a 2D Riemannian manifold with constant sectional curvature K, congruent triangles have equal angles, so:



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Motivating the uniqueness and existence of roots

For us to use the root of $f(x) = \cos(c\sqrt{x}) - \cos(a\sqrt{x})\cos(b\sqrt{x})$ as a definition for a sectional curvature estimate, we have to show it has exactly two zeroes in $x \in \left(-\infty, \frac{\pi^2}{\max(a,b,c)^2}\right]$ (one being the trivial $x = 0$ root), unless $a^2 + b^2 = c^2$ in which case $x = 0$ is the only (double) root.

1. For negative x, we can use the power mean inequality to show order-by-order that that there can only be one relevant root
2. For positive x it is non-trivial, and only strong numerical evidence is currently available

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Hard Annulus Random Geometric Graphs

Definition 3.1

A *random geometric graph* of a metric space (X, d) with respect to a connection function $I : \mathbb{R} \to [0, 1]$ is a graph obtained by taking the vertices to be a random (uniform w.r.t. the volume measure) sample of N points $V \subset X$, with the probability of two vertices $v_1, v_2 \in V$ being connected given by $I(d(v_1, v_2))$.

Definition 3.2

A *hard annulus random geometric graph* with connection length ℓ and tolerance $0 < p < 1$ is a random geometric graph with connection function

$$I(d) = \begin{cases} 1 & |d - \ell| < \ell p \\ 0 & \text{otherwise} \end{cases}$$

Metric distortion: Background

We want a measure of how good or bad an embedding of a graph into another metric space is. Such notions are called metric distortions, and a recurring useful definition in their study is:

Definition 3.3

The *embedding ratio* between two points $u, v \in G$ under an embedding between metric spaces $f : G \to X$ is defined as

$$\rho_f(u,v) = \frac{d_X(f(u), f(v))}{d_G(u,v)} \qquad u, v \in G.$$

We can use this to define a notion of distortion, that satisfies the following properties:

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We can use this to define a notion of distortion, that satisfies the following properties:

1. Manifest invariance to inverting definition of embedding ratios
2. Rescaling either metric should be irrelevant.

Metric distortion: solution

A natural definition of our metric distortion is to take the mean deviation of the logarithm of the ratios.

Definition 3.4

For a finite metric space G, we define the metric distortion of an embedding $f : G \to X$ to a metric space X to be

$$\text{dist}(f) = \frac{1}{|G|^2} \sum_{u,v \in G} |\log(\rho_f(u,v)) - \log(\overline{\rho_f}_{\text{geom}})|,$$

where the geometric mean is given by

$$\overline{\rho_f}_{\text{geom}} = \left[\prod_{u,v \in G} \rho_f(u,v) \right]^{|G|^{-2}}.$$

Metric distortion: induced edge length

Definition 3.5

For a graph G with the usual combinatorial metric and an embedding $f : G \to X$ into another metric space, the *effective edge length* induced by f is

$$\ell_e = \overline{\rho_f}_{\text{geom}}$$

such that we scale $\rho'_f(u,v) = \frac{\rho_f(u,v)}{\overline{\rho_f}_{\text{geom}}}$, simplifying the expression for the distortion of a graph with the effective edge length induced by f to

$$\text{dist}(f) = \frac{1}{|G|^2} \sum_{u,v \in G} |\log(\rho'_f(u,v))|.$$

Construction of manifold-like graphs

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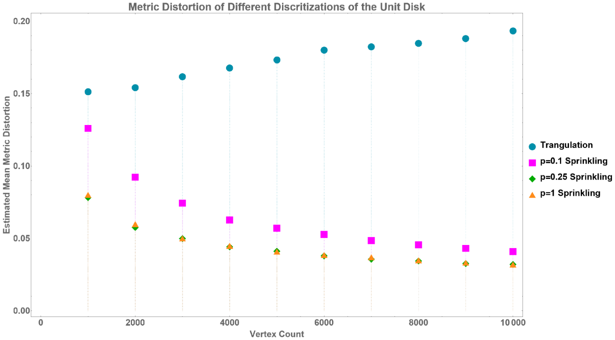
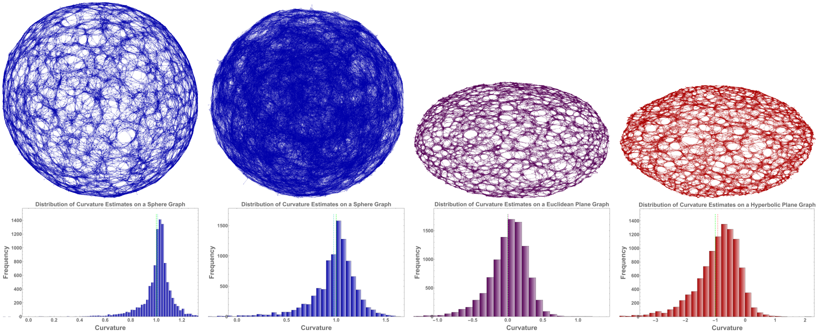
2. Choose tolerance $p = 0.25$ as heuristic for low edge count and distortion
3. Use binary search to find minimal connection length ℓ giving a connected hard annulus random geometric graph



Sectional curvature distributions of large ($\sim 10^4$ vertices) 'constant curvature' graphs



S^2	S^3	Euclidean disk	Hyperbolic disk
1	1	0	-1
1.0088 ± 0.0013	0.974 ± 0.0027	-0.04 ± 0.16	-0.997 ± 0.028

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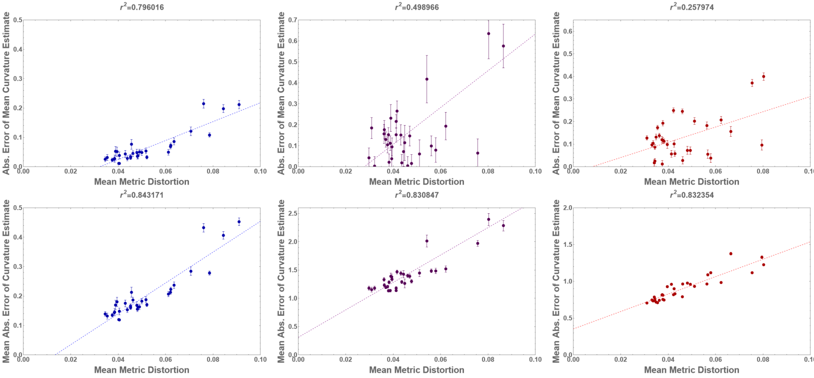
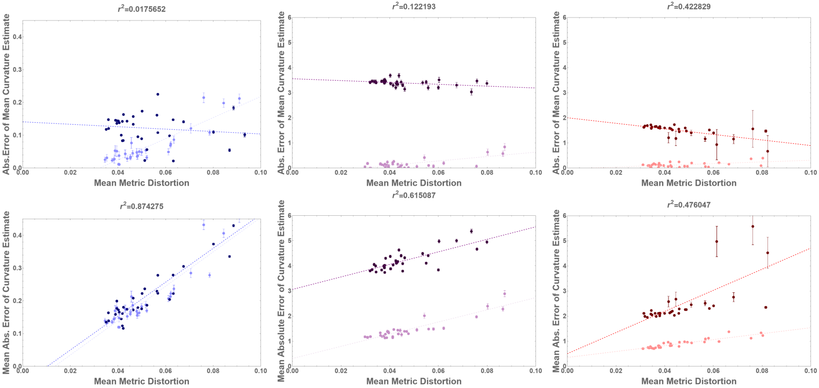
Apparent convergence of mean curvature

We can take a variety of tolerances and vertex counts to get a range of metric distortion graphs, in order to see if the deviation of the curvature distributions mean from (and the mean deviation from) the underlying mean correlates with the distortion.



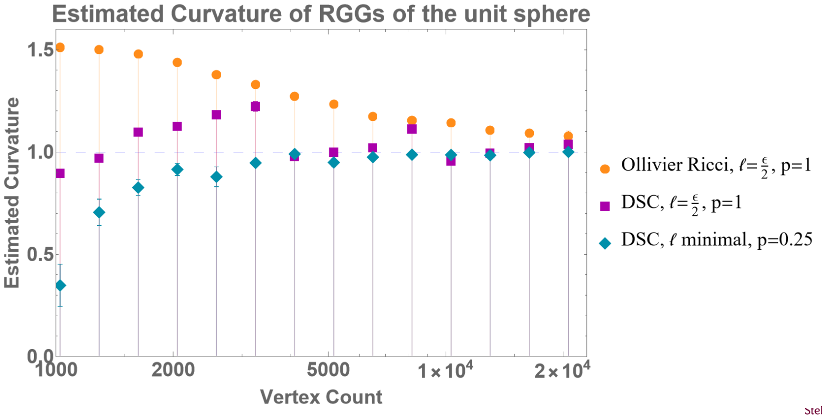
Comparison: Wolfram-Ricci curvature

We can compare with Wolfram-Ricci curvature using the same test:



Comparison: Mesoscopic Ollivier-Ricci curvature

Comparing with a convergence test done on a modified definition of Ollivier-Ricci curvature:



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Triangles on the earth

We can draw right-angled triangles on the earth! Therefore, we can calculate radius of curvature $R = \frac{1}{\sqrt{K}}$ estimates for constructed triangles:

The earth has an average radius of ≈ 6371 km, and from 100 triangles we get an estimate of (6370.5 ± 0.7) km.

Estimating the radius distribution of the earth

It is possible to derive the distribution of radii of an oblate spheroid with parameters corresponding to the earth

$$p(R) \propto (R - 6357)^{-\frac{1}{2}} \qquad 6357 \leq R \leq 6399$$

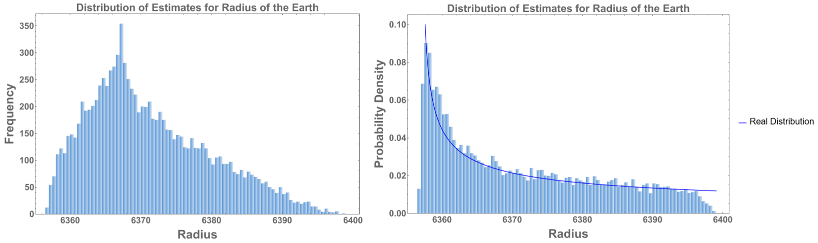

Figure: The left plot shows the frequency of different estimated radii over 10^4 samples. The right plot shows the probability density of different radius estimates over 10^4 with a maximum length scale of 6400 km as compared to the expected probability density for a oblate spheroid with parameters comparable to that of earth.

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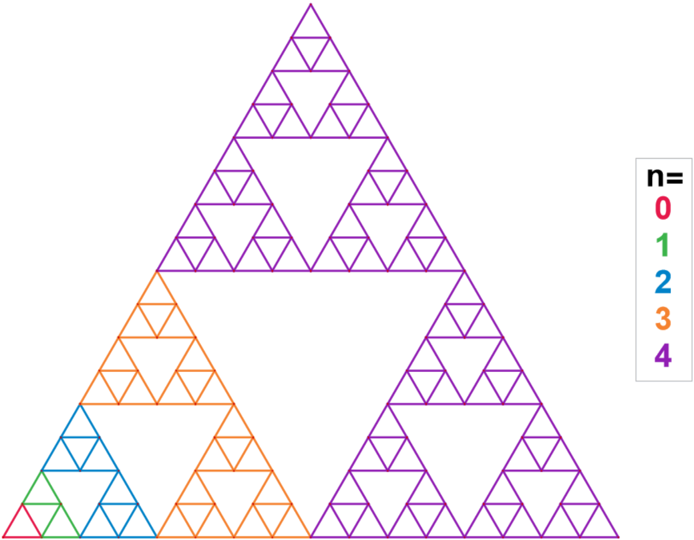
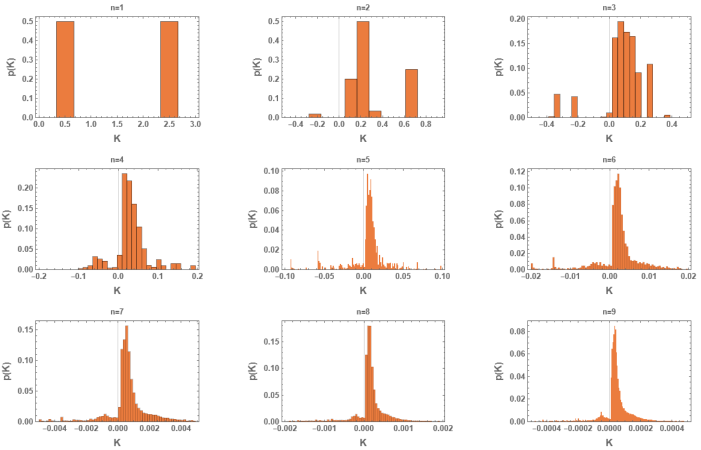
Sierpinski triangle graphs $\hat{S}_3^n$



Sectional curvature distributions of Sierpinski triangle graphs



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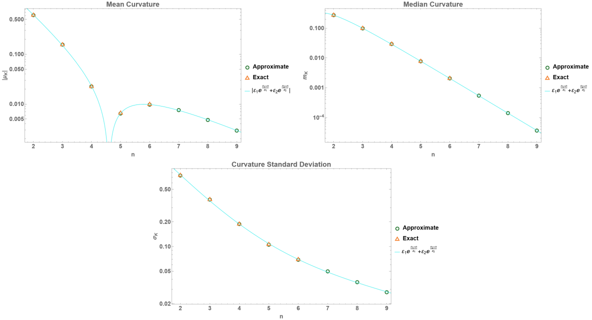
Scaling of statistics and limiting distribution



When looking at the scaling of the mean, median and standard deviation on log plots, one notices the apparent empirical law $\varepsilon_1 e^{\frac{n_1-n}{a_1}} + \varepsilon_2 e^{\frac{n_2-n}{a_2}}$ appears to model all three statistics with different constants, functions with fitted constants plotted above.

Scaling of sectional curvature distribution

Since $\varepsilon_1 e^{\frac{n_1-n}{a_1}}$ carries the large n behaviour of the empirical law, we can see that if for each n we scaled edge lengths by $\ell_{\text{conv}} = e^{-\frac{1}{2a_1}}$, the corresponding statistic's empirical law would converge to some finite value $\varepsilon_1 e^{\frac{n_1}{a_1}}$. The fitted constants and their corresponding convergence lengths are given in the table below:

	ε_1	n_1	a_1	ℓ_{conv}
μ	-1	$-1.23 \pm 0.0.32$	1.78 ± 0.06	0.755 ± 0.007
M	$+1$	1.41 ± 0.06	0.743 ± 0.007	0.5102 ± 0.0032
σ	$+1$	-8 ± 9	4.8 ± 2.4	0.90 ± 0.05

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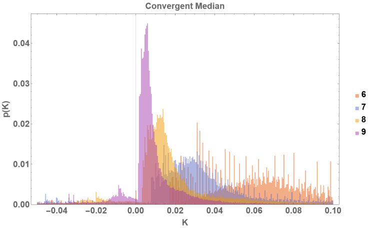
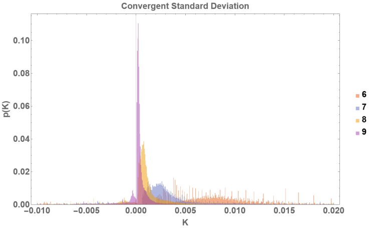
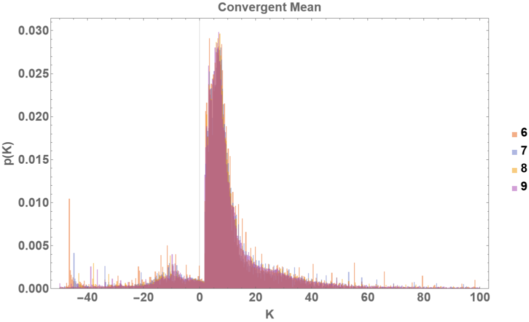
Scaling of sectional curvature distribution

We can now look at the rescaled distributions







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Conclusions

- We introduced a new notion of metric distortion
- Hard annulus random geometric graphs gives an efficient way to construct low distortion graphs
- We defined a sectional curvature estimator, applicable to both continuous and discrete geometries that seems algorithmically and computationally simpler than tested alternatives
- We validated our approach using random geometric graphs constructed from different manifolds, and showed that our estimator produces accurate and convergent results, outperforming tested alternatives

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Conclusions

- We can accurately estimate the radius of the earth and distribution of curvature of an oblate spheroid, showing it works outside of just constant sectional curvature
- We can apply sectional curvature distributions to Sierpinski triangle graphs and find non-trivial behaviour
- Has potential implications for quantum gravity, numerical relativity, data science, computational geometry and more